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Characterizing 4-critical graphs with Ore-degree at most seven

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A R T I C L E I N F O

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A B S T R A C T

Dirac introduced the notion of a k-*critical* graph, a graph that is not $(k-1)$-colorable but whose every proper subgraph is $(k-1)$-colorable. Brook's Theorem states that every graph with maximum degree k is k-colorable unless it contains a subgraph isomorphic to K_{k+1} or an odd cycle (for $k=2$). Equivalently, for all $k \geq 4$, the only k-critical graph of maximum degree $k-1$ is K_k. A natural generalization of Brook's theorem is to consider the *Ore-degree* of a graph, which is the maximum of $d(u)+d(v)$ over all $uv \in E(G)$. Kierstead and Kostochka proved that for all $k \geq 6$ the only k-critical graph with Ore-degree at most $2k-1$ is K_k. Kostochka, Rabern and Stiebitz proved that the only 5-critical graphs with Ore-degree at most 9 are K_5 and a graph they called O_5.

A different generalization of Brook's theorem, motivated by Hajos' construction, is Ore's conjectured bound on the minimum density of a k-critical graph. Recently, Kostochka and Yancey proved Gallai's conjecture. Their proof for $k \geq 5$ implies the above results on Ore-degree. However, the case for $k = 4$ remains open, which is the subject of this paper.

Kostochka and Yancey's short but beautiful proof for the case $k = 4$ says that if G is a 4-critical graph, then $|E(G)| \geq (5|V(G)|-2)/3$. We prove the following bound which is better when there exists a large independent set of degree three vertices: if G is a 4-critical graph G, then $|E(G)| \geq 1.6|V(G)| + .2\alpha(D_3(G)) - .6$, where $D_3(G)$ is the graph

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induced by the degree three vertices of G. As a corollary, we characterize the 4-critical graphs with Ore-degree at most seven as precisely the graphs of Ore-degree seven in the family of graphs obtained from K_4 and Ore compositions.

1. Introduction

All graphs considered in this paper are simple and finite. Graph coloring is an important area of study in graph theory.

1.1. Ore-degree

We know that the chromatic number of a graph is at most the maximum degree plus one. It is natural to ask for what graphs does the equality hold. Since chromatic number and maximum degree are both monotone properties, it suffices to consider the minimal non-colorable graphs.

We say a graph is *k-critical* if G is not $(k-1)$-colorable but every proper subgraph is.

Brooks [1] proved the following theorem characterizing the k-critical graphs with maximum degree $k-1$:

Theorem 1.1 *(Brooks). For all $k \geq 4$, the only k-critical graph with maximum degree $k-1$ is K_k. In addition, the only 3-critical graphs with maximum degree two are odd cycles.*

An interesting question is to ask whether Brooks' theorem can be improved. One manner in which to ask this question is to consider the maximum degree of edges instead of vertices. To that end, we define the *Ore-degree* of a graph as the maximum of $d(u)+d(v)$ for every edge $uv \in E(G)$. Brook's Theorem says that for all $k \geq 4$ if G is a k-critical graph with Ore-degree at most $2k-2$, then G is isomorphic to K_k. Kostochka and Kierstead [4] extended Brooks' theorem to graphs with Ore-degree at most $2k-1$ for all $k \geq 6$ as follows.

Theorem 1.2 *(Kostochka and Kierstead). For all $k \geq 6$, the only k-critical graph with Ore-degree at most $2k-1$ is K_k.*

This is best possible since C_5 join K_{k-3} is k-critical but has Ore-degree $2k$. Note that the only 3-critical graphs are odd cycles. The remaining cases to consider are $k=4$ and $k=5$. Settling the case $k=5$, Kostochka, Rabern and Stiebitz [6] characterized the 5-critical graphs with Ore-degree at most 9. There are only two such graphs, K_5 and

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