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A new proof of the flat wall theorem

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ABSTRACT

We give an elementary and self-contained proof, and a numerical improvement, of a weaker form of the excluded clique minor theorem of Robertson and Seymour, the following. Let $t, r \geq 1$ be integers, and let $R = 49152t^{24}(40t^2 + r)$. An r-wall is obtained from a $2r \times r$-grid by deleting every odd vertical edge in every odd row and every even vertical edge in every even row, then deleting the two resulting vertices of degree one, and finally subdividing edges arbitrarily. The vertices of degree two that existed before the subdivision are called the pegs of the r-wall. Let G be a graph with no K_t minor, and let W be an R-wall in G. We prove that there exist a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$ in the following sense. There exists a separation (X, Y) of $G - A$ such that $X \cap Y$ is a subset of the vertex set of the cycle C' that bounds the outer face of W', $V(W') \subseteq Y$, every peg of W' belongs to X and the graph $G[Y]$ can almost be drawn in the unit disk with the vertices $X \cap Y$ drawn on the boundary of the disk in the order determined by C'. Here almost means that the assertion holds after repeatedly removing parts of the graph separated from $X \cap Y$ by a cutset Z of size at most three, and adding all edges with both ends in Z. Our proof

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gives rise to an algorithm that runs in polynomial time even when r and t are part of the input instance. The proof is self-contained in the sense that it uses only results whose proofs can be found in textbooks.

1. Introduction

All graphs in this paper are finite, and may have loops and parallel edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An H *minor* is a minor isomorphic to H. There is an ever-growing collection of so-called excluded minor theorems in graph theory. These are theorems which assert that every graph with no minor isomorphic to a given graph or a set of graphs has a certain structure. The best known such theorem is perhaps Wagner's reformulation of Kuratowski's theorem [17], which says that a graph has no K_5 or $K_{3,3}$ minor if and only if it is planar. One can also characterize graphs that exclude only one of those minors. To state such a characterization for excluded K_5 we need the following definition. Let H_1 and H_2 be graphs, and let J_1 and J_2 be complete subgraphs of H_1 and H_2, respectively, with the same number of vertices. Let G be obtained from the disjoint union of $H_1 - E(J_1)$ and $H_2 - E(J_2)$ by choosing a bijection between $V(J_1)$ and $V(J_2)$ and identifying the corresponding pairs of vertices. We say that G is a *clique-sum* of H_1 and H_2. Since we allow parallel edges, the set that results from the identification of $V(J_1)$ and $V(J_2)$ may include edges of the clique-sum. For instance, the graph obtained from K_4 by deleting an edge can be expressed as a clique-sum of two smaller graphs, where one is a triangle and the other is a triangle with a parallel edge added. By V_8 we mean the graph obtained from a cycle of length eight by adding an edge joining every pair of vertices at distance four in the cycle. The characterization of graphs with no K_5 minor, due to Wagner [16], reads as follows.

Theorem 1.1. *A graph has no K_5 minor if and only if it can be obtained by repeated clique-sums, starting from planar graphs and V_8.*

There are many other similar theorems; a survey can be found in [3]. Theorem 1.1 is very elegant, but attempts at extending it run into difficulties. For instance, no characterization is known for graphs with no K_6 minor, and there is evidence suggesting that such a characterization would be fairly complicated. Even if a characterization of graphs with no K_6 is found, there is no hope in finding one for excluding K_t for larger values of t.

Thus when excluding an H minor for a general graph H we need to settle for a less ambitious goal—a theorem that gives a necessary condition for excluding an H minor,

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