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Decompositions of graphs into cycles with chords

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In memory of Dick Schelp, who passed away shortly after the submission of this paper

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ABSTRACT

We show that if G is a graph on at least $3r+4s$ vertices with minimum degree at least $2r+3s$, then G contains $r+s$ vertex disjoint cycles, where each of s of these cycles either contain two chords, or are of order 4 and contain one chord.

1. Introduction and main result

The following beautiful conjecture of Bialostochi, Finkel, and Gyárfás appears in [1].

Conjecture 1. *Let r, s be nonnegative integers and let G be a graph with $|V(G)| \geq 3r+4s$ and minimum degree $\delta(G) \geq 2r+3s$. Then G contains a collection of r cycles and s chorded cycles, all vertex disjoint.*

The complete bipartite graph $K_{2r+3s-1,n-2r-3s+1}$ shows that the minimum degree cannot be lowered when $n \geq 4r+6s-2$.

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The conjecture is a generalization of well known results of Pósa, and of Corrádi and Hajnal. Pósa proved (see [7, problem 10.2]) that any graph with minimum degree at least 3 contains a chorded cycle and Corrádi and Hajnal [3] proved that any graph of minimum degree at least $2r$ of order $n \geq 3r$ contains r vertex disjoint cycles.

The purpose of this article is to show that a stronger result than that given in the conjecture is true. We prove the following theorem.

Theorem 2. *If G is a simple graph on $|V(G)| \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then G contains $r + s$ vertex disjoint cycles, each of s of them either with two chords, or a C_4 with one chord.*

It is likely the case that among the chorded "long" cycles more than two chords will be present, and that one can insist on two chords even in the C_4s, but our method of proof does not establish this.

It has come to our attention that Conjecture 1 has been proved by Gao, Li, and Yan and appears in [4], but they do not address the stronger result given in our theorem. Also a degree sum condition is used by Chiba, Fujita, Gao, and Li in [2], and neighborhood union conditions are used by Gao, Li, and Yan [5], and by Qiao [8], to realize disjoint chorded cycles. Finally, independently of our results, Gould, Hirohata, and Horn [6] proved a result on the existence of disjoint doubly chorded cycles under a degree sum condition. This result however only applies to the $r = 0$ case with $|V(G)| \geq 6s$.

The proof of our theorem is based on several technical theorems and lemmas, the last two of which are in themselves of special interest. One of these (Theorem 12) generalizes the result of Pósa mentioned earlier by showing that a graph with minimum degree 3 contains a cycle with two chords.

We write as usual P_n, C_n, K_n, or E_n for a path, cycle, complete graph, or empty graph respectively on n vertices. When the number of vertices is unspecified we shall write for example P_* or C_*. We denote by C_n^{+k} any cycle of length n with at least k additional chords, and K_n^{-k} the complete graph on n vertices with at most k edges removed. If $k = 1$ then we write just C_n^+ or K_n^- for brevity. Note that, for example, a C_n^{+3} graph is also considered as a special case of a C_n^{+2} graph. It will also be convenient to denote by $C_*^\dagger$ a graph that is *either* a C_*^{+2} *or* a C_4^+. We shall write $G \cup H$ for the vertex disjoint union of G and H, and $G \cup H + ke$ for such a graph with k additional edges added *between* G and H. We shall also use the notation $H \subseteq G$ or $G \supseteq H$ to indicate that H is a subgraph of G, and $H \subset G$ or $G \supset H$ to indicate that H is a non-spanning subgraph of G, i.e., a subgraph with $|V(H)| < |V(G)|$. For example, the statement $G \supset C_*$ indicates that G contains a non-hamiltonian cycle. We shall occasionally abuse notation by regarding a subgraph $H \subseteq G$ also as a subset of vertices of G. So for example, we write $G[H \cup v]$ instead of $G[V(H) \cup \{v\}]$ for the subgraph induced by the vertices of H and an extra vertex $v \in V(G)$.

The proof of Theorem 2 involves a number of technical theorems and lemmas, the relevance of which only becomes apparent in the proof of Theorem 2. Thus we shall first