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An excluded minors method for infinite matroids

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ABSTRACT

The notion of thin sums matroids was invented to extend the notion of representability to non-finitary matroids. A matroid is tame if every circuit–cocircuit intersection is finite. We prove that a tame matroid is a thin sums matroid over a finite field k if and only if all its finite minors are representable over k.

1. Introduction

Given a family of vectors in a vector space over some field k, there is a matroid structure on that family whose independent sets are given by the linearly independent subsets of the family. Matroids arising in this way are called *representable* matroids over k. A classical theorem of Tutte [14] states that a finite matroid is binary (that is, representable over $\mathbb{F}_2$) if and only if it does not have $U_{2,4}$ as a minor. In the same spirit, a key aim of finite matroid theory has been to determine such 'forbidden minor' characterisations for the classes of matroids representable over other finite fields. For example Bixby and Seymour [2,13] characterised the finite ternary matroids (those representable over $\mathbb{F}_3$) by forbidden minors, and more recently there is a forbidden minors characterisation for the

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finite matroids representable over $\mathbb{F}_4$, due to Geelen, Gerards and Kapoor [10]. In 1971 Rota conjectured that for any finite field the class of finite matroids representable over that field is characterised by finitely many forbidden minors. A proof of this conjecture has been announced by Geelen, Gerards and Whittle. An outline of the proof has already appeared in [11]. In this paper we develop a method which makes it possible to extend the above excluded minor characterisations from finite to infinite matroids.

It is clear that any representable matroid is *finitary*, that is, all its circuits are finite, and so many interesting examples of infinite matroids are not representable. However, since the construction of many standard examples, including the algebraic cycle matroids of infinite graphs, is suggestively similar to that of representable matroids, the notion of *thin sums matroids* was introduced in [7]: it is a generalisation of representability which captures these infinite examples. We will work with thin sums matroids rather than with representable matroids.

In [1] it was shown that the class of tame thin sums matroids over a fixed field is closed under duality, where a matroid is *tame* if any circuit–cocircuit intersection is finite. On the other hand, there are thin sums matroids whose dual is not a thin sums matroid [4] – such counterexamples cannot be tame. A simple consequence of this closure under duality is that the class of tame thin sums matroids over a fixed field is closed under taking minors, and so we may consider the forbidden minors for this class.

Minor closed classes may have infinite 'minimal' forbidden minors. For example the class of finitary matroids has the infinite circuit $U^*_{1,\mathbb{N}}$ as a forbidden minor. Similarly, the class of tame thin sums matroids over $\mathbb{R}$ has $U_{2,\mathcal{P}(\mathbb{R})}$ as a forbidden minor. However, our main result states that the class of tame thin sums matroids over a fixed *finite* field has only finite minimal forbidden minors.

Theorem 1.1. *Let M be a tame matroid and k be a finite field. Then M is a thin sums matroid over k if and only if every finite minor of M is k-representable.*

The proof is by a compactness argument. All previous compactness proofs in infinite matroid theory known to the authors use only that either all finite restrictions or all finite contractions have a certain property to conclude that the matroid itself has the desired property. For our purposes, arguments of this kind must fail because there is a tame matroid all of whose finite restrictions and finite contractions are binary but which is not a thin sums matroid over $\mathbb{F}_2$ – in fact, it has a $U_{2,4}$-minor. We shall briefly sketch how to construct such a matroid. Start with $U_{2,4}$, and add infinitely many elements parallel to one of its elements. This ensures that every finite contraction is binary. If we also add infinitely many elements which are parallel in the dual to some other element then we guarantee in addition that all finite restrictions are binary, but the matroid itself has a $U_{2,4}$ minor.

Theorem 1.1 implies that each of the excluded minor characterisations for finite representable matroids mentioned in the first paragraph extends to tame matroids. Thus, for example, a tame matroid is a thin sums matroid over $\mathbb{F}_2$ if and only if it has no $U_{2,4}$ mi-

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