Journal of Combinatorial Theory, Series B ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Partitioning H-minor free graphs into three subgraphs with no large components

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ARTICLE INFO

Article history:
Received 29 March 2015
Available online xxxx

Keywords:
Vertex partitions
Graph minors
Odd minors
Small components

ABSTRACT

We prove that for every graph H, if a graph G has no (odd) H minor, then its vertex set $V(G)$ can be partitioned into three sets X_1, X_2, X_3 such that for each i, the subgraph induced on X_i has no component of size larger than a function of H and the maximum degree of G. This improves a previous result of Alon, Ding, Oporowski and Vertigan (2003) [1] stating that $V(G)$ can be partitioned into four such sets if G has no H minor. Our theorem generalizes a result of Esperet and Joret (2014) [9], who proved it for graphs embeddable on a fixed surface and asked whether it is true for graphs with no H minor.

As a corollary, we prove that for every positive integer t, if a graph G has no K_{t+1} minor, then its vertex set $V(G)$ can be partitioned into $3t$ sets $X_1, \ldots, X_{3t}$ such that for each i, the subgraph induced on X_i has no component of size larger than a function of t. This corollary improves a result of Wood (2010) [21], which states that $V(G)$ can be partitioned into $\lceil 3.5t + 2\rceil$ such sets.

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[1] Partially supported by National Science Foundation under Grant No. DMS-1664593.
[2] Supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2017R1A2B4005020).

http://dx.doi.org/10.1016/j.jctb.2017.08.003
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1. Introduction

The famous Four Color Theorem states that every planar graph G admits a partition of its vertex set into four sets X_1, X_2, X_3, X_4 such that for $1 \leq i \leq 4$, every component of the subgraph $G[X_i]$ induced on X_i has at most one vertex. Certainly there are planar graphs whose vertex set cannot be partitioned into three such sets. However, Esperet and Joret [9] proved that the number of sets can be reduced to three, if we relax each X_i to induce a subgraph having no component of size larger than a function of the maximum degree of G.

Theorem 1.1 *(Esperet and Joret [9]). Let Σ be a surface of Euler genus g. If a graph G is embeddable on Σ and has maximum degree at most Δ, then $V(G)$ can be partitioned into three sets X_1, X_2, X_3 such that for $1 \leq i \leq 3$, every component of $G[X_i]$ has at most $(5\Delta)^{2^g-1}(15\Delta)^{(32\Delta+8)2^g}$ vertices.*

The number of sets in Theorem 1.1 is best possible, since a $k \times k$ triangular grid has maximum degree six but its vertex set cannot be partitioned into two sets such that each set induces a subgraph with no component of size less than k by the famous HEX lemma [10]. In contrast, Alon, Ding, Oporowski, and Vertigan [1] showed that for graphs of bounded tree-width and bounded maximum degree, it is possible to partition the vertex set into two sets inducing subgraphs having no large components.

Theorem 1.2 *(Alon et al. [1, Theorem 2.2][3]). Let $w \geq 3$ and Δ be positive integers. If a graph G has maximum degree at most Δ and tree-width at most w, then $V(G)$ can be partitioned into X_1, X_2 such that for $1 \leq i \leq 2$, every component of $G[X_i]$ has at most $24w\Delta$ vertices.*

It was pointed out by Esperet and Joret [private communication, 2015] that the condition of maximum degree mentioned in Theorem 1.2 cannot be removed. See Theorem 4.1 for details.

Though it is impossible to partition all planar graphs of bounded maximum degree into two induced subgraphs with components of bounded size, it is possible to partition them such that the tree-width of every component is small. More precisely, DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan [3] proved the following result, which was conjectured by Thomas [19]. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

Theorem 1.3 *(DeVos et al. [3]). For every graph H, there exists an integer N such that if H is not a minor of G, then $V(G)$ can be partitioned into two sets X_1, X_2 such that the tree-width of $G[X_i]$ is at most N for $1 \leq i \leq 2$.*

[3] In [1], Theorem 1.2 is stated without requiring $w \geq 3$. However, [1] cites [5, (3.7)], which requires $w \geq 3$. However, Theorem 1.2 is true even if $w < 3$, because a stronger statement was proved by Wood [20].

Please cite this article in press as: C.-H. Liu, S. Oum, Partitioning H-minor free graphs into three subgraphs with no large components, J. Combin. Theory Ser. B (2017), http://dx.doi.org/10.1016/j.jctb.2017.08.003