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# Circumference of 3-connected cubic graphs

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A B S T R A C T

The circumference of a graph is the length of its longest cycles. Jackson established a conjecture of Bondy by showing that the circumference of a 3-connected cubic graph of order $n$ is $\Omega(n^{0.694})$. Bilinski et al. improved this lower bound to $\Omega(n^{0.753})$ by studying large Eulerian subgraphs in 3-edge-connected graphs. In this paper, we further improve this lower bound to $\Omega(n^{0.8})$. This is done by considering certain 2-connected cubic graphs, finding cycles through two given edges, and distinguishing the cases according to whether or not these edges are adjacent.



## 1. Introduction

Tait [10] conjectured in 1880 that every 3-connected cubic planar graph contains a Hamilton cycle. This conjecture remained open until a counterexample was found in

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1946 by Tutte [11]. There has since been extensive research concerning longest cycles in graphs, see [6] for more references. We use $|G|$ to denote the *order* of a graph $G$, i.e., the number of vertices in $G$; and refer to the length of a longest cycle in $G$ as the *circumference* of $G$. We will be concerned with lower bounds on the circumference of 3-connected cubic graphs.

Barnette [4] showed that every 3-connected cubic graph of order $n$ has circumference $\Omega(\log n)$. Bondy and Simonovits [7] improved this bound to $\exp(\Omega(\sqrt{\log n}))$ and conjectured that it can be improved further to $\Omega(n^c)$ for some constant $0 < c < 1$. This conjecture was confirmed by Jackson [8], with $c = \log_2(1 + \sqrt{5}) - 1 \approx 0.694$. Bondy and Simonovits [7] constructed an infinite family of 3-connected cubic graphs with circumference $\Theta(n^{\log_9 8}) \approx \Theta(n^{0.946})$.

Recently, Bilinski, Jackson, Ma and Yu [6] showed that every 3-connected cubic graph of order $n$ has circumference $\Omega(n^\alpha)$, where $\alpha \approx 0.753$ is the real root of $4^{1/x} - 3^{1/x} = 2$. This is proved by reducing the problem to one about Eulerian subgraphs in 3-edge-connected graphs.

In this paper, we further improve this lower bound by considering certain vertex weighted, 2-connected cubic graphs (multiple edges allowed). Let $G$ be a graph and let $w : V(G) \to \mathbb{Z}^+$, where here $\mathbb{Z}^+$ denotes the set of non-negative integers. For any $H \subseteq G$, we write $w(H) := \sum_{v \in V(H)} w(v)$.

**Theorem 1.1.** *Let $r = 0.8$ and $c = 1/(8^r - 6^r) \approx 0.922$. Let $G$ be a 2-connected cubic graph, let $w : V(G) \to \mathbb{Z}^+$, and let $e, f \in E(G)$. Suppose every 2-edge cut in $G$ separates $e$ from $f$. Then there is a cycle $C$ in $G$ with $e, f \in E(C)$ such that*

*(a) $w(C) \geq w(G)^r$ when $e$, $f$ are adjacent, and*
*(b) $w(C) \geq cw(G)^r$ when $e$, $f$ are not adjacent.*

**Remark 1.2.** We may assume that the weight function $w$ in Theorem 1.1 satisfies $w(v) = 0$ for every vertex $v$ incident with $e$ or $f$.

To see this, we define a new weight function $w' : V(G) \to \mathbb{Z}^+$ such that $w'(v) = 0$ if $v$ is incident with $e$ or $f$, and $w'(v) = w(v)$ otherwise. Let $w_0 = \sum w(v)$, where the sum is over all vertices incident with $e$ or $f$. Then $w(G) = w'(G) + w_0$. If Theorem 1.1 holds for $w'$ then there is a cycle $C$ such that $e, f \in E(C)$, $w'(C) \geq w'(G)^r$ (when $e$, $f$ are adjacent), and $w'(C) \geq cw'(G)^r$ (when $e$, $f$ are not adjacent). Thus $w(C) = w'(C) + w_0$, $w(C) \geq (w(G) - w_0)^r + w_0 \geq w(G)^r$ (when $e$, $f$ are adjacent), and $w(C) \geq c(w(G) - w_0)^r + w_0 \geq cw(G)^r$ (when $e$, $f$ are not adjacent). Thus, Theorem 1.1 also holds for $w$.

In our proof of Theorem 1.1 we divide $G$ into a few smaller parts, find long cycles in some of these parts, and merge these cycles into the desired cycle in $G$. The length of the cycle will be guaranteed by various properties of the function $x^r$ (see Lemma 2.4). We will need structural information of graphs obtained from a 3-connected cubic graph after certain operations, and we will also need cycles through some specified edges and