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Two-regular subgraphs of odd-uniform hypergraphs ☆

Jie Han, Jaehoon Kim

School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom

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ABSTRACT

Let $k \geq 3$ be an odd integer and let n be a sufficiently large integer. We prove that the maximum number of edges in an n-vertex k-uniform hypergraph containing no 2-regular subgraphs is $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$, and the equality holds if and only if H is a full k-star with center v together with a maximal matching omitting v. This verifies a conjecture of Mubayi and Verstraëte.

1. Introduction

Turán problems are central in extremal graph theory. In general, Turán-type problems ask for the maximum number of edges of a (hyper)graph that does not contain certain subgraph(s). Their generalizations to hypergraphs appear to be extremely hard – for example, despite many existing works, the Turán density of a tetrahedron (four triples on four vertices) is still unknown (see [8]).

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E-mail addresses: J.Han@bham.ac.uk (J. Han), j.kim.3@bham.ac.uk (J. Kim).

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Erdős [3] asked to determine the maximum size $f_k(n)$ of an n-vertex k-uniform hypergraph without any generalized 4-cycles, i.e., four distinct edges A, B, C, D such that $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$. For $k = 2$, this reduces to a well-known problem of finding the Turán number for the 4-cycle. It is known that $f_2(n) = (1 + o(1))n^{3/2}$ [2,4] and the exact value of $f_2(n)$ for infinitely many n is obtained in [6]. For $k \geq 3$, Füredi [7] showed that $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor \leq f_k(n) \leq \frac{7}{2}\binom{n}{k-1}$ [1] and conjectured the following.

Conjecture 1.1. *For $k \geq 4$ and $n \in \mathbb{N}$, $f_k(n) = \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$.*

The lower bound is achieved by a full k-star together with a maximal matching omitting its center. Here a *full k-star* is a k-uniform n-vertex hypergraph which consists of all $\binom{n-1}{k-1}$ sets of size k containing a given vertex v, and the given vertex v is called the *center* of the full k-star. The most recent result on $f_k(n)$ is due to Pikhurko and Verstraëte [13], who showed that $f_k(n) \leq \min\{1 + 2/\sqrt{k}, 7/4\}\binom{n}{k-1}$, and $f_3(n) \leq \frac{13}{9}\binom{n}{2}$. This improves a result by Mubayi and Verstraëte [11]. In [9], the second author made a related conjecture about k-uniform hypergraphs containing no r pairs of disjoint sets with the same union when k is sufficiently bigger than r.

Since the generalized 4-cycles are 2-regular, i.e., each vertex has degree 2, one way to relax the original problem of Erdős is to consider the maximum size of n-vertex (hyper)graphs without any 2-regular sub(hyper)graphs (or more generally, without any r-regular subgraphs). In fact, the (relaxed) problem is of its own interest even for graphs. Pyber [14] proved that the largest number of edges in a graph with no r-regular subgraphs is $O(n \log n)$ for any $r \geq 2$, and in [15], Pyber, Rödl and Szemerédi showed that there are graphs with no r-regular subgraphs having $\Omega(n \log \log n)$ edges for any $r \geq 3$.

For non-uniform hypergraphs, it is easy to see that any hypergraph with no r-regular subgraphs has at most $2^{n-1} + r - 2$ edges and Kostochka and the second author [10] showed that if $n \geq \max\{425, r + 1\}$ then any n-vertex hypergraph with no r-regular subgraphs having the maximum number of edges must contain a vertex of degree 2^{n-1}. For uniform hypergraphs, the problem becomes more interesting. One natural candidate for the extremal example of k-uniform hypergraphs with no 2-regular subgraphs is the full k-star. Indeed, Mubayi and Verstraëte [12] proved the following.

Theorem 1.2. *([12]) For every even integer $k \geq 4$, there exists n_k such that the following holds for all $n \geq n_k$. If H is an n-vertex k-uniform hypergraph with no 2-regular subgraphs, then $|H| \leq \binom{n-1}{k-1}$. Moreover, equality holds if and only if H is a full k-star.*

In [9] the second author generalized the arguments in [12] and showed similar results for k-uniform hypergraphs with no r-regular subgraphs when $r \in \{3, 4\}$. Moreover, for

[1] In fact, Füredi [7] found a slightly better lower bound for $k = 3$, namely, $f_3(n) \geq \binom{n}{2}$ for $n \equiv 1$ or 5 mod 20.