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The square of a planar cubic graph is 7-colorable

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ABSTRACT

We prove the conjecture made by G. Wegner in 1977 that the square of every planar, cubic graph is 7-colorable. Here, 7 cannot be replaced by 6.

1. Introduction

We prove the conjecture made by G. Wegner [11] in 1977, mentioned by Gionfriddo [4] and listed in the monograph by Jensen and Toft [6], that the square of every planar, cubic graph is 7-colorable. To see that this bound is best possible, consider first the cubic prism graph with six vertices. Then subdivide an edge which is not contained in a triangle. The square of this graph is a complete graph with seven vertices. Now we take two copies of this graph and add an edge between them so that we obtain a cubic graph. This cubic graph is planar and its square has chromatic number 7.

The proof is based on a decomposition method: We color the vertices of the planar, cubic graph by two colors, red and blue, such that the blue square-graph is 3-colorable, and the red square-graph is planar and hence 4-colorable, by the 4-Color Theorem.

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Wegner's 7-color conjecture proved in the present paper is part of a more general problem on the chromatic number of squares of planar graphs. After submission of the present paper a number of papers have been written on this subject, see e.g. [1], [2], [3], [7], [10] and the references in these papers. A computer aided proof of the 7-color theorem has recently been obtained in [5].

2. Terminology and notation

The terminology is the same as in [6] and [9].

A *k-path* is a path with k vertices. A *k-cycle* is defined analogously.

In a plane embedding of a connected graph every face boundary is a walk called a *facial walk*. A *facial path* is a path which is a subgraph of a facial walk. If C is a cycle in a plane graph, then *the interior of* C, denoted $int(C)$, consists of the edges and vertices inside C. Thus, an edge joining a vertex in C with a vertex inside C is also in $int(C)$. Sometimes $int(C)$ also refers to a graph, namely the subgraph of G induced by the vertices inside C. The precise meaning will always be clear from the context.

If G is a graph, then the *square* G^2 of G is obtained from G by adding all edges joining vertices of distance 2 in G. If we color the vertices of G red or blue, then the *red subgraph* (or just the *red graph*) is the subgraph of G induced by the red vertices. The *red square-subgraph* (or just the *red square-graph*) is the subgraph of G^2 induced by the red vertices. Similar notation is used for the blue vertices.

If some vertices of G are colored $1, 2, 3$ such that the coloring is proper in G^2, then we say that *vertex* v *can see color* i if there is a vertex u of color i such that u is a neighbor of v in G^2. A *Kempe chain with colors* i, j is a connected component in the subgraph of G^2 induced by the vertices of colors i, j.

We shall also use the following notation: If we have already named a sequence $v_1, v_2, \ldots$ of vertices in the cubic graph and that sequence includes say two neighbors of v_1, then the neighbor of v_1 which is not in the list is called *the third neighbor of* v_1. If precisely one neighbor of v_2 is in the list, then the two neighbors of v_2 not in the list are called *the two other neighbors of* v_2.

3. Decomposing the vertex set of a cubic graph

We shall now indicate the idea in the proof of Wegner's conjecture. We begin with a conjecture.

Conjecture 1. *If G is a 3-connected, cubic graph, then the vertices of G can be colored blue and red such that the blue subgraph has maximum degree 1 (that is, it consists of a matching and some isolated vertices) and the red subgraph has minimum degree at least 1 and contains no 4-path.*