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Notes

The maximum number of cliques in graphs without long cycles

Ruth Luo [1]

University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA

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ABSTRACT

The Erdős–Gallai Theorem states that for $k \geq 3$ every graph on n vertices with more than $\frac{1}{2}(k-1)(n-1)$ edges contains a cycle of length at least k. Kopylov proved a strengthening of this result for 2-connected graphs with extremal examples $H_{n,k,t}$ and $H_{n,k,2}$. In this note, we generalize the result of Kopylov to bound the number of s-cliques in a graph with circumference less than k. Furthermore, we show that the same extremal examples that maximize the number of edges also maximize the number of cliques of any fixed size. Finally, we obtain the extremal number of s-cliques in a graph with no path on k-vertices.

1. Introduction

In [4], Erdős and Gallai determined $ex(n, P_k)$, the maximum number of edges in an n-vertex graph that does not contain a copy of the path on k vertices, P_k. This result was a corollary of the following theorem:

E-mail address: ruthluo2@illinois.edu.

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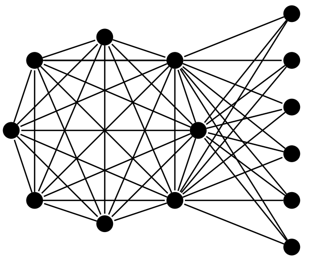

Fig. 1. $H_{14,11,3}$.

Theorem 1.1 *(Erdős and Gallai [4]). Let G be an n-vertex graph with more than $\frac{1}{2}(k-1)(n-1)$ edges, $k \geq 3$. Then G contains a cycle of length at least k.*

To obtain the result for paths, suppose G is an n-vertex graph with no copy of P_k. Add a new vertex v adjacent to all vertices in G, and let this new graph be G'. Then G' is an $n+1$-vertex graph with no cycle of length $k+1$ or longer, and so $e(G)+n = e(G') \leq \frac{1}{2}kn$ edges.

Corollary 1.2 *(Erdős and Gallai [4]). Let G be an n-vertex graph with more than $\frac{1}{2}(k-2)n$ edges, $k \geq 2$. Then G contains a copy of P_k.*

Both results are sharp with the following extremal examples: for Theorem 1.1, when $k-2$ divides $n-1$, take any connected n-vertex graph whose blocks (maximal connected subgraphs with no cut vertices) are cliques of order $k-1$. For Corollary 1.2, when $k-1$ divides $n-1$, take the n-vertex graph whose connected components are cliques of order $k-1$.

There have been several alternate proofs and sharpenings of the Erdős–Gallai theorem including results by Woodall [15], Lewin [13], Faudree and Schelp[6,5], and Kopylov [12] – see [8] for further details.

The strongest version was that of Kopylov who improved the Erdős–Gallai bound for 2-connected graphs. To state the theorem, we first introduce the family of extremal graphs.

Fix $k \geq 4$, $n \geq k$, $\frac{k}{2} > a \geq 1$. Define the n-vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets A, B, C such that $|A| = a$, $|B| = n-k+a$ and $|C| = k-2a$ and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$. (See Fig. 1.)

Note that when $a \geq 2$, $H_{n,k,a}$ is 2-connected, has no cycle of length k or longer, and $e(H_{n,k,a}) = \binom{k-a}{2} + (n-k+a)a$.

Definition. Let $f_s(n,k,a) := \binom{k-a}{s} + (n-k+a)\binom{a}{s-1}$, where $f_2(n,k,a) = e(H_{n,k,a})$.

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