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Non-trivial non weakly pseudocompact spaces

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ABSTRACT

A space Z is weakly pseudocompact if Z is G_δ-dense in at least one of its compactifications. In 1996 F.W. Eckertson [3] proposed the following problem: Find examples of Baire non Lindelöf spaces which are not weakly pseudocompact. Eckertson proposed a list of natural candidates. In this article we show that part of this list produces examples of this type by providing examples of product spaces which are Baire non-Lindelöf and not weakly pseudocompact.

1. Introduction

The notion of weak pseudocompactness, introduced in [5], is a natural generalization of a well known characterization of pseudocompactness. A space X is *weakly pseudocompact* if it is G_δ-dense in some of its compactifications.

On the one hand, every pseudocompact space is weakly pseudocompact and weak pseudocompactness is a productive property which provide us with abundant examples of weakly pseudocompact spaces. On the other hand, every weakly pseudocompact space is Baire, and weakly pseudocompact Lindelöf spaces

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are compact [5]; in other words, non Baire spaces and Lindelöf noncompact spaces are not weakly pseudocompact. Following Eckertson [3], we say that a space X is *trivially non weakly pseudocompact* if either X is Lindelöf noncompact or X is not Baire. Eckertson proved, for example, that no countable power of the Sorgenfrey line is weakly pseudocompact. But, as he pointed out, the knowledge of weakly pseudocompact spaces suffers from the scarcity of non trivial examples of non weakly pseudocompact spaces. He provided a list of spaces that "obviously" should not be weakly pseudocompact. However, as was mentioned in [10], proving weak pseudocompactness or its absence for some individual spaces turns out to be a surprisingly difficult task for some very simple spaces.

One of the spaces proposed by Eckertson is the one-point extension $D_\lambda = D \cup \{\infty\}$ of a discrete space D, where $\omega < \lambda < |D|$, topologized as follows: the points of D are isolated and the neighborhoods of ∞ are of the form $\{\infty\} \cup A$, where $A \subset D$ and $|D \setminus A| \leq \lambda$. Yet, if $\lambda^\omega = \lambda$ the authors of [10] constructed a compactification of D_λ where this space is G_δ-dense. Hence, contrary to Eckertson's expectations, this kind of spaces are in many cases weakly pseudocompact.

Another interesting kind of spaces proposed by Eckertson are the products of uncountably many Baire non-Lindelöf spaces; the cases of ω^T, $\mathbb{R}^T$ and $\mathbb{S}^T$ are particularly intriguing, where $\mathbb{S}$ is the Sorgenfrey line. We have been not able to answer the case of these particular spaces, but we clarify the situation when $X = \prod_{t\in T} X_t$ is the product of an uncountable family of Lindelöf Σ-spaces with countable π-weight and which do not admit a dense Čech-complete subspace, and when $G = \prod_{t\in T} G_t$ is the product of an uncountable family of Lindelöf Σ non Čech-complete topological groups. We show that, in both cases, these products and each one of their dense subspaces, whose projections cover all countable faces, are never weakly pseudocompact. We apply these results for some particular spaces providing examples of product spaces which are non-trivially non weakly pseudocompact spaces, as is the case of uncountable powers of Bernstein subsets of the real line. It is worth mentioning that, in the case of topological groups, some of these examples also can be obtained applying the results in [2].

After the Introduction this paper is organized as follows: Section 2 below is devoted to prove some technical results about the Stone–Čech compactification of some topological products which we use in Section 3 to prove our main results. Lastly, Section 4 provides concrete examples of non-trivial non weakly pseudocompact spaces.

As usual, the real line with the Euclidean topology is denoted by $\mathbb{R}$, and its subspace of the natural numbers will be denoted by $\mathbb{N}$. The symbol ω denotes the first infinite cardinal number. The first uncountable cardinal number is ω_1. For a space X and a subset A of X, $\mathrm{cl}_X A$ ($\mathrm{int}_X A$) will mean the closure (interior) of A in the space X. If there is no doubt as to what space X we are considering, we will simply write $\mathrm{cl}\, A$ ($\mathrm{int}\, A$) instead of $\mathrm{cl}_X A$ ($\mathrm{int}_X A$). The statement *"$X \subset Y$ is G_δ-dense in Y"* means that each nonempty G_δ-set in Y contains at least one point in X. For a set Y, $[Y]^{\leq\omega}$ and $[Y]^{<\omega}$ signify the collection of all countable subsets and of all finite subsets of Y, respectively. A *compactification* of a space X is a compact space K containing a copy of X as a dense subspace. βX denotes the Stone–Čech compactification of X. A subset N of X is *nowhere dense* in X if each nonempty open subset of X contains a nonempty open subset which misses N. A subset of X is *meager* in X if it is a countable union of nowhere dense subsets of X. A space X is *Baire* if every countable family of open dense subsets of X has a dense intersection. A collection $\mathcal{B}$ of open sets (sets with nonempty interior) in X is said to be a *π-base* (*π-pseudobase*) for X if each nonempty open subset of X contains a member of $\mathcal{B}$. A space X is *Oxtoby-complete* (*Todd-complete*) if there is a sequence $\{\mathcal{B}_n\}_{n<\omega}$ of π-bases (π-pseudobases) in X such that for any sequence $\{U_n\}_{n<\omega}$ satisfying $U_n \in \mathcal{B}_n$ and $\mathrm{cl}_X U_{n+1} \subset \mathrm{int}_X U_n$, for all $n < \omega$, we have that $\bigcap_{n<\omega} U_n \neq \emptyset$.

Throughout this article all topological spaces are considered Tychonoff.