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# The pass move is an unknotting operation for welded knots ☆

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ABSTRACT

It is known that the pass move is not an unknotting operation in classical knot theory. In this paper, we prove that the pass move is an unknotting operation in welded knot theory.



## 1. Introduction

A pass move is a local deformation on oriented diagrams as shown in Fig. 1. In classical knot theory, it is known that any classical knot can be deformed into the trivial knot or the trefoil knot by a finite sequence of pass moves, which are distinguished by the Arf invariant.

The notion of a welded knot was introduced in [2] as a generalization of a classical knot. The set of welded knots contains that of classical knots properly. There is a relation between a welded knot and a knotted torus in 4-space [12].

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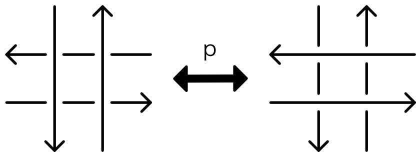


**Fig. 1.** A pass move.

In this paper, we consider the pass move for welded knot diagrams such that we allow the intermediates in the process to be not only classical but also welded knot diagrams. Then we have the following.

**Theorem 1.1.** *Any welded knot can be deformed into the trivial knot by a finite sequence of pass moves.*

This theorem induces that even if a classical knot has the Arf invariant one, it can be deformed into the trivial knot by a finite sequence of pass moves in welded sense. Therefore, it is natural to ask the question whether there are infinitely many classical knots of Arf invariant one by a single pass move or not. Then we have the following.

**Theorem 1.2.** *There are infinitely many classical knots $K$ such that*

(i) *the Arf invariant of $K$ is equal to one, and*
(ii) *$K$ can be deformed into the trivial knot by a single pass move.*

For a $\mu$-component welded link $L = \bigcup_{1 \leq i \leq \mu} K_i$, the $(i, j)$-*linking number* of $L$ $(1 \leq i \neq j \leq \mu)$ is the sum of the signs of $(i, j)$-crossings, and denoted by $\lambda_{ij}(L)$. Here, an $(i, j)$-*crossing* of $L$ is a crossing of a diagram where the over- and under-paths belong to $K_i$ and $K_j$, respectively.

**Theorem 1.3.** *For two $\mu$-component welded links $L$ and $L'$ with $\mu \geq 2$, $L$ can be deformed into $L'$ by a finite sequence of pass moves if and only if*

(i) $\lambda_{ij}(L) - \lambda_{ji}(L) = \lambda_{ij}(L') - \lambda_{ji}(L')$ *for any* $1 \leq i < j \leq \mu$, *and*
(ii) $\sum_{\substack{1 \leq s \leq \mu \\ s \neq i}} \lambda_{is}(L) \equiv \sum_{\substack{1 \leq s \leq \mu \\ s \neq i}} \lambda_{is}(L') \pmod 2$ *for any* $1 \leq i \leq \mu$.

A sharp move [7] is an unknotting operation for classical knots, which is coincident with a pass move ignoring the orientations. In [1,13], a sharp move is also shown to be an unknotting operation for welded knots. The following theorem implies that the sharp move is equivalent to the pass move for welded links.

**Theorem 1.4.** *For two $\mu$-component welded links $L$ and $L'$, the following are equivalent.*

(i) *$L$ can be deformed into $L'$ by a finite sequence of pass moves.*
(ii) *$L$ can be deformed into $L'$ by a finite sequence of sharp moves.*

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2 by unknotting the knots constructed by Murakami [8] by a single pass move. In Section 4, we give the complete classification of welded links up to pass moves (Theorem 1.3). In Section 5, we study a relationship between pass and sharp moves, and prove Theorem 1.4.