Topology and its Applications 247 (2018) 41–49

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The quasi-Rothberger property of linearly ordered spaces

Zuquan Li

Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, PR China [1]

ARTICLE INFO

Article history:
Received 26 June 2017
Received in revised form 8 June 2018
Accepted 20 July 2018
Available online 23 July 2018

MSC:
primary 54D20
secondary 54F05

Keywords:
Quasi-Lindelöf
Quasi-Rothberger
Extent
Anti-well-ordered
Linearly ordered space

ABSTRACT

A space X is said to be quasi-Rothberger if for each closed set $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X, there is a $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $F \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$. In this article, we give necessary and sufficient conditions of the quasi-Rothberger property of linearly ordered spaces.

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1. Introduction

A.V. Arhangel'skiǐ [1] introduced the quasi-Lindelöf property in connection with the estimation of the cardinality of topological spaces. A space X is said to be *quasi-Lindelöf* if for every closed set $F \subset X$ and each cover $\mathcal{U}$ of F by open subsets of X, there is a countable family $\{U_n : n \in \mathbb{N}\}$ of $\mathcal{U}$ such that $F \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$. The quasi-Lindelöf property is weaker than the Lindelöf property and stronger than the weakly Lindelöf property.

Recently, G. Di Maio and Lj.D.R. Kočinac [4] defined the following quasi-Rothberger property as a selective version of the quasi-Lindelöf property:

Definition 1.1. A space X is said to be *quasi-Rothberger* if for each closed set $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X, there is a $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $F \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set X, the following selection principles are defined:

E-mail address: hzsdlzq@sina.com.
[1] Current address.

https://doi.org/10.1016/j.topol.2018.07.011
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$\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection principle: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{\mathsf{fin}}(\mathcal{A},\mathcal{B})$ denotes the selection principle: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ such that B_n is a finite subset of A_n for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

Let $\mathcal{O}$ denote the collection of all the open covers of X, then $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ is called the Rothberger property and $\mathsf{S}_{\mathsf{fin}}(\mathcal{O},\mathcal{O})$ is known as the Menger property [2, Table 1].

For a space X and a subset F of X, we write:

- $\mathcal{O}_F = \{\mathcal{U} : \mathcal{U}$ is a cover of F by sets open in $X\}$;
- $\mathcal{O}_F^D = \{\mathcal{U} : \mathcal{U}$ is a family of open subsets of X such that $F \subset \overline{\bigcup \mathcal{U}}\}$.

Thus a space X is quasi-Rothberger if and only if X satisfies $\mathsf{S}_1(\mathcal{O}_F, \mathcal{O}_F^D)$ for each closed subset F of X.

Recall that a space X is said to be *weakly Lindelöf* [4] if for each open cover $\mathcal{U}$ of X, there is a $U_n \in \mathcal{U}$ for each $n \in \mathbb{N}$ such that $X = \overline{\bigcup_{n\in\mathbb{N}} U_n}$. A space X is said to be *weakly Rothberger* [3] if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there is a $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $X = \overline{\bigcup_{n\in\mathbb{N}} U_n}$.

We have the following implications.

$$\begin{tikzcd}
\text{Rothberger} \arrow[r] \arrow[d] & \text{quasi-Rothberger} \arrow[r] \arrow[d] & \text{weakly Rothberger} \arrow[d] \\
\text{Lindelöf} \arrow[r] & \text{quasi-Lindelöf} \arrow[r] & \text{weakly Lindelöf}
\end{tikzcd}$$

Example 1.2. There exists a quasi-Lindelöf space which is not quasi-Rothberger.

Proof. Let $\beta\mathbb{N} \setminus \mathbb{N}$ be the remainder of Čech–Stone compactification of $\mathbb{N}$, then $\beta\mathbb{N} \setminus \mathbb{N}$ satisfies $\mathsf{S}_{\mathsf{fin}}(\mathcal{O},\mathcal{O})$ since it is compact. Thus $\beta\mathbb{N}\setminus\mathbb{N}$ is Lindelöf. Moreover, $\beta\mathbb{N}\setminus\mathbb{N}$ is quasi-Lindelöf. By Lemma 2.5 in [2], $\beta\mathbb{N}\setminus\mathbb{N}$ is not weakly Rothberger. So $\beta\mathbb{N} \setminus \mathbb{N}$ is not quasi-Rothberger. □

Note that there are no implications between the Lindelöf property and the quasi-Rothberger property. Example 1.2 implies that there is a Lindelöf space which is not quasi-Rothberger.

Example 1.3 *([6, Example 14.8]).* There exists a quasi-Rothberger space which is not Lindelöf.

Proof. Let $\mathbb{R}$ be with usual topology τ, $X = \{x_\alpha : 0 \le \alpha < \omega_1\}$ is a subset of $\mathbb{R}$ of cardinality of ω_1, where ω_1 denotes the first uncountable ordinal. For each $\alpha < \omega_1$, take $V_\alpha = \{x_\beta : 0 \le \beta \le \alpha\}$. Denote

$$\tau' = \{V \cap X : V \in \tau\} \bigcup \{V_\alpha : 0 \le \alpha < \omega_1\}.$$

Then τ' is a subbase for a topology of X. X is a hereditarily separable T_2 space. Therefore, X is quasi-Rothberger by [4, Proposition 2.2]. But X is not Lindelöf. □

There are very few papers which deal with the quasi-Rothberger. In [4] G. Di Maio and Lj.D.R. Kočinac obtained the following result:

Proposition 1.4 *([4, Proposition 2.2]). Every hereditarily separable space X is quasi-Rothberger.*

On the other hand, hereditary separability is not necessary for a space being quasi-Rothberger. The following example illustrates that there exists a quasi-Rothberger space which is not even separable.

Example 1.5. There exists a quasi-Rothberger space which is not separable.

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