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Ribbon crossing numbers, crossing numbers, and Alexander polynomials

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ABSTRACT

A 2-knot is a surface in $\mathbf{R^4}$ that is homeomorphic to $\mathbf{S^2}$, the standard sphere in 3-space. A ribbon 2-knot is a 2-knot obtained from m 2-spheres in $\mathbf{R^4}$ by connecting them with $m-1$ annuli. Let K^2 be a ribbon 2-knot. The ribbon crossing number, denoted by $r\text{-}cr(K^2)$ is a numerical invariant of the ribbon 2-knot K^2. It is known that the degree of the Alexander polynomial of K^2 is less than or equal to $r\text{-}cr(K^2)$. In this paper, we show that $r\text{-}cr(K^2)$ is estimated by coefficients in the Alexander polynomial of K^2. Furthermore, applying this fact, for a classical knot k^1, we also estimate the crossing number, denoted by $cr(k^1)$.

1. Introduction

We start out with a few definitions from previous papers. A *ribbon 2-knot* is a sphere in $\mathbf{R^4}$ obtained from m spheres by connecting them with $m-1$ annuli. Let K^2 be a ribbon 2-knot. We call a set of spheres and annuli to construct K^2 a *ribbon presentation* of K^2. For a ribbon presentation of K^2 we call the sum of the number of times that annuli cross spheres the *ribbon crossing number* of the ribbon presentation. We also call the smallest ribbon crossing number of all ribbon presentations of K^2 the *ribbon crossing number* of K^2, denoted by $r\text{-}cr(K^2)$. In this paper, we are interested in estimating the ribbon crossing number of a ribbon 2-knot K^2 in $\mathbf{R^4}$ and the crossing number of a classical knot k^1 in $\mathbf{R^3}$.

Concerning classical knots, the following fact is known well; see [1, p. 132].

Fact 1.1. *Let $\Delta_{k^1}(t)$ be the Alexander polynomial of a classical knot k^1, $deg\Delta_{k^1}(t)$ its degree, and $cr(k^1)$ the crossing number of k^1. Then the following holds.*

$$1 + deg\Delta_{k^1}(t) \leq cr(k^1), \tag{1}$$

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where $deg\Delta_{k^1}(t)$ is the difference of between the maximal degree and the minimal degree of $\Delta_{k^1}(t)$.

Concerning ribbon 2-knots, we showed the following proposition; see, [2, Theorem 1.1].

Proposition 1.2. *Let $\Delta_{K^2}(t)$ be the Alexander polynomial of a ribbon 2-knot K^2, $deg\Delta_{K^2}(t)$ its degree, and $r\text{-}cr(K^2)$ the ribbon crossing number of K^2. Then the following holds.*

$$deg\Delta_{K^2}(t) \leq r\text{-}cr(K^2), \tag{2}$$

where $deg\Delta_{K^2}(t)$ is the difference of between the maximal degree and the minimal degree of $\Delta_{K^2}(t)$.

Note that in [2] and [3] we called the ribbon crossing number in this paper the crossing number of K^2 and denoted it by $cr(K^2)$.

In Sect. 3, we give proofs of Theorem 1.3 and Corollary 1.4. Then we also give their applications.

Theorem 1.3. *Let K^2 be a ribbon 2-knot, $r\text{-}cr(K^2)$ the ribbon crossing number of K^2. If $r\text{-}cr(K^2) = n$, then there exists some representative $\Delta_{K^2}(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$ of the Alexander polynomial of K^2 in its equivalence class up to units such that $|a_p| \leq {}_nC_p$ $(p = 0, 1, \cdots, n)$.*

Let U be the set of units in $\mathbf{Z}[t, t^{-1}]$. For instance, let $spun(5_2)$ be the spun 2-knot of the classical knot 5_2 in the Alexander-Briggs table. We can naturally construct a ribbon presentation with ribbon crossing number 4; see, [2, Theorem 4.1]. Then we have that $r\text{-}cr(spun(5_2)) \leq 4$. As the ribbon 2-knot $spun(5_2)$ has the Alexander polynomial $2 - 3t + 2t^2$ $(mod\ U)$, we have that $2 \leq r\text{-}cr(spun(5_2))$ by Proposition 1.2. On the other hand, we have that $4 \leq r\text{-}cr(spun(5_2))$ by Theorem 1.3. Therefore, $r\text{-}cr(spun(5_2)) = 4$.

The following corollary is an immediate consequence of Theorem 1.3 and its proof. Let K^2 be an arbitrary ribbon 2-knot and $|a_p|$ the absolute value of a_p in Theorem 1.3 $(p = 0, 1, \cdots, n)$. Then we have the following corollary.

Corollary 1.4.

$$log_2(\Sigma_{p=0}^{n}|a_p| + 1) \leq r\text{-}cr(K^2) \tag{3}$$

In Sect. 4, we give a proof of the following theorem.

Theorem 1.5. *Let k^1 be a classical knot, $cr(k^1)$ the crossing number of k^1, and $\Delta_{k^1}(t)$ the Alexander polynomial of k^1. If $cr(k^1) = n$, then there exists some representative $\Delta_{k^1}(t) = a_0 + a_1t + a_2t^2 + \cdots + a_{n-1}t^{n-1}$ of the Alexander polynomial of k^1 in its equivalence class up to units such that $|a_p| \leq {}_{n-1}C_p$ $(p = 0, 1, \cdots, n-1)$.*

For instance, if a classical knot k^1 has the Alexander polynomial $2 - 3t + 2t^2$ $(mod\ U)$, we find that $5 \leq cr(k^1)$ by Theorem 1.5. Note that 5_2-knot has this Alexander polynomial. From Fact 1.1, we only have that $3 \leq cr(k^1)$.

The following corollary is an immediate consequence of Theorem 1.5 and its proof. Let k^1 be an arbitrary classical knot and $|a_p|$ the absolute value of a_p in Theorem 1.5 $(p = 0, 1, \cdots, n-1)$.

Corollary 1.6.

$$1 + log_2(\Sigma_{p=0}^{n-1}|a_p| + 1) \leq cr(k^1) \tag{4}$$