Topology and its Applications 247 (2018) 100–114

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Crossing numbers of random two-bridge knots[☆]

Moshe Cohen[a], Chaim Even-Zohar[b,*], Sunder Ram Krishnan[c]

[a] Department of Mathematics and Statistics, Vassar College, Poughkeepsie, NY, 12604, USA
[b] Department of Mathematics, University of California, Davis, CA, 95616, USA
[c] Andrew and Erna Viterbi Faculty of Electrical Engineering, Technion – Israel Institute of Technology, Haifa 32000, Israel

ARTICLE INFO

Article history:
Received 1 June 2016
Received in revised form 1 August 2018
Accepted 2 August 2018
Available online 7 August 2018

MSC:
57M25
57M27
60C05
60B99

Keywords:
Bertrand's Ballot theorem
Binary runs
Binary words
Chebyshev knot
Random topology
Rational knot

ABSTRACT

In a previous work, the first and third authors studied a random knot model for all two-bridge knots using billiard table diagrams. Here we present a closed formula for the distribution of the crossing numbers of such random knots. We also show that the probability of any given knot appearing in this model decays to zero at an exponential rate as the length of the billiard table goes to infinity. This confirms a conjecture from the previous work.

1. Introduction

There is a recent resurgence of interest in the study of random knots. Several new random models focus on diagrams for which knot invariants can be calculated more easily. The random model in this work is a billiard trajectory with randomly chosen crossings, as introduced in [8] by the first and third authors.

☆ The first and third authors were supported in part by the funding from the European Research Council under Understanding Random Systems via Algebraic Topology (URSAT), ERC Advanced Grant 320422. The second author was supported by BSF 2012188.

* Corresponding author.
E-mail addresses: mcohen@vassar.edu (M. Cohen), chaim@math.ucdavis.edu (C. Even-Zohar), eeksunderram@gmail.com (S.R. Krishnan).

https://doi.org/10.1016/j.topol.2018.08.001
0166-8641/

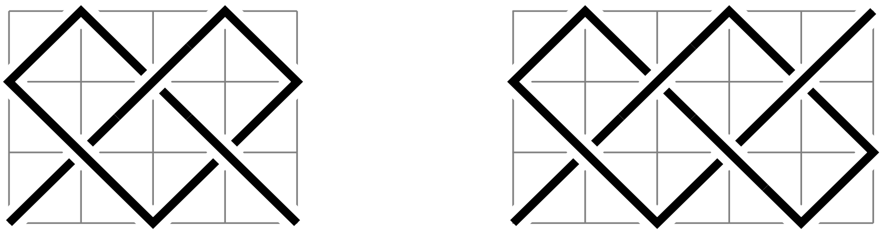

Fig. 1. Billiard table diagrams for the trefoil and the figure-eight knots.

The second author studies other random knot models in [13] and in work with Hass, Linial, and Nowik [14] using petal diagrams of Adams et al. [1] and random grid diagrams. Other random knot models can be found in work by Dunfield, Obeidin et al. [11], by Cantarella, Chapman and Mastin [6], and by Westenberger [32]. Previous models for random knotting include the closures of braids obtained from random walks in braid groups, and, more prominently, the knotting of random walks in three dimensions. This last setting was featured in the textbook "Random knotting and linking" edited by Millet and Sumners [24]. The reader can find more specific references in [14] and [8].

We remark that some of the above models exhibit *local knotting*, in the sense that a typical prime connected summand can be attributed to a small part of the curve that represents the knot [28,26,10,31,23,7]. Other models, including the one we study here, seem to avoid this kind of localization. In this respect, it is intersting to compare our model to lattice knots in thin tubes studied by Beaton et al. [3,4] and Ishihara et al. [16].

Denote by $T(a,b)$ the planar trajectory of a billiard ball in $[0,a] \times [0,b]$ fired at slope one from the lower left corner, where a and b are coprime integers. *Billiard table diagrams* are obtained from these curves by deciding the crossing information and connecting the two ends outside. See Fig. 1 for billiard table diagrams of the trefoil and the figure-eight knots, with $a = 3$ for both and $b = 4$ and 5, respectively.

The billiard trajectory $T(a,b)$ is plane isotopic to the curve given as $x(t) = \cos at$ and $y(t) = \cos bt$, where $t \in [0,\pi]$. Such oscillating curves appeared in several natural constructions of knots as parametrized paths in three dimensions. *Harmonic knots* are obtained by closing the open path $(\cos at,\ \cos bt,\ \cos ct)$, where $t \in [0,\pi]$ and a,b,c are pairwise coprime integers [9]. *Lissajous knots* are parametrized by the already-closed curve $(\cos(at+\phi), \cos(bt+\theta), \cos(ct+\psi))$ where $t \in [0,2\pi]$, and $\phi,\theta,\psi \in \mathbb{R}$ are fixed phase shifts [5,17,15,2].

Not all knots are harmonic or Lissajous. However, the diagrams resulting from the projection of Lissajous curves to the xy-plane give rise to all knots by suitable choice of the crossing information. This was shown by Lamm [22] in the study of *Fourier knots*.

Koseleff and Pecker [21] prove a similar statement for the *open* harmonic path $(\cos at, \cos bt)$, in their work on *Chebyshev knots*. They reparametrize it as $(T_a(t), T_b(t))$ using the Chebyshev polynomials $T_n(\cos\theta) = \cos n\theta$, and show that any choice for the crossings can be realized by some $z(t) = T_c(t+\phi)$. Vassiliev represents all knots by polynomials in [30].

The first and third authors of the present paper turn these constructions into a model for random knots [8] by deciding the crossing information {⤫, ⤬}^{n} with coin flips independently and uniformly at each crossing of $T(a,b)$. Here we continue with the study of random knots with bridge number at most two, obtained by setting $a = 3$ and letting $b = n+1$ where $n = 0$ or 1 mod 3. We denote by D_n a random knot diagram in this model and by K_n the resulting random knot.

Much of the literature on random knots focuses on the probability of obtaining a given knot. We progress to study the distribution of the random variable $\mathrm{c}(K_n)$, the crossing number of a random knot. Indeed, it is natural to ask how the number of crossings n in a random billiard table diagram D_n relates to the minimal number of crossings in any diagram of the resulting knot K_n. In this respect, we prove the following main result.