Topology and its Applications 246 (2018) 69–82

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A method of returning vector-valued maps to real-valued functions on monotone operators ☆

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ARTICLE INFO

Article history:
Received 21 September 2017
Accepted 3 July 2018
Available online 9 July 2018

MSC:
54C08
54D20
54E20
46A40

Keywords:
Ordered topological vector space
Positive interior point
Minkowski functional
Gauge
Interior of the positive cone
Monotonically countably paracompact
Monotonically countably metacompact
Stratifiable
Semi-staratifiable
Monotone cb-space
Perfectly normal
Monotone operator

ABSTRACT

Let Y be an ordered topological vector space with a positive interior point e. Motivated by the Minkowski functional p_e, we introduce the function q_e on the interior $\mathrm{int}_Y Y^+$ of the positive cone of Y. The function q_e plays a role for evaluating a 'reverse gauge' of the Minkowski functional p_e on the restricted space $\mathrm{int}_Y Y^+$. We also apply the function q_e for giving complete proof of the recent results [7] by Jin, Xie and Yue on monotonically countably paracompact spaces. New characterizations by vector-valued continuous maps are also given for stratifiable spaces and monotone cb-spaces, the latter provides a positive answer to a question posed in [12] by the author.

1. Introduction

Throughout this paper, let $\mathbb{R}$ be the set of all real numbers, and $\mathbb{N}$ the set of all natural numbers. All topological spaces are assumed to be Hausdorff spaces.

☆ This work was supported by TCUE Special Grant-in-Aid (A) 2017.
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https://doi.org/10.1016/j.topol.2018.07.004
0166-8641/

For two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ of subsets of a space, it is written that $(A_n) \preccurlyeq (B_n)$ if $A_n \subset B_n$ for each $n \in \mathbb{N}$. A topological space X is said to be *monotonically countably metacompact* ([6]) if there exists an operator U assigning to each decreasing sequence $(D_j)_{j\in\mathbb{N}}$ of closed subsets of X with $\bigcap_{j\in\mathbb{N}} D_j = \emptyset$, a sequence $U((D_j)) = (U(n,(D_j)))_{n\in\mathbb{N}}$ of open subsets of X such that (1) $D_n \subset U(n,(D_j))$ for each $n \in \mathbb{N}$; (2) $\bigcap_{n\in\mathbb{N}} U(n,(D_j)) = \emptyset$; (3) If $(D_j) \preccurlyeq (E_j)$, then $U((D_j)) \preccurlyeq U((E_j))$. If, in addition, (2′) $\bigcap_{n\in\mathbb{N}} \overline{U(n,(D_j))} = \emptyset$ holds, X is said to be *monotonically countably paracompact* ([6]).

The following theorem was obtained by C. Good, R. Knight and I. Stares [6, Theorem 25], C. Good and L. Haynes [5, Theorem 3] and the author [11, Corollary 3.3]. The symbol $(0,\infty)$ stands for the set $\{r \in \mathbb{R} : r > 0\}$.

Theorem 1.1 *([6], [5], [11]). For a topological space X, the following statements (a), (b), (c) and (d) are mutually equivalent to X being monotonically countably paracompact.*

- (a) *There exists an operator Φ assigning to each locally upper-bounded function $f : X \to \mathbb{R}$, a locally upper-bounded l.s.c. function $\Phi(f) : X \to \mathbb{R}$ with $f \le \Phi(f)$ such that $\Phi(f) \le \Phi(f')$ whenever $f \le f'$.*
- (b) *There exist operators Φ and Ψ assigning to each u.s.c. function $f : X \to \mathbb{R}$, an l.s.c. function $\Phi(f) : X \to \mathbb{R}$ and a u.s.c. function $\Psi(f) : X \to \mathbb{R}$ with $f \le \Phi(f) \le \Psi(f)$ such that $\Phi(f) \le \Phi(f')$ and $\Psi(f) \le \Psi(f')$ whenever $f \le f'$.*
- (c) *There exist operators Φ and Ψ assigning to each l.s.c. function $f : X \to (0,\infty)$, a u.s.c. function $\Phi(f) : X \to (0,\infty)$ and an l.s.c. function $\Psi(f) : X \to (0,\infty)$ with $\Psi(f) \le \Phi(f) \le f$ such that $\Phi(f) \le \Phi(f')$ and $\Psi(f) \le \Psi(f')$ whenever $f \le f'$.*
- (d) *There exists an operator Φ assigning to each locally bounded function $f : X \to \mathbb{R}$, a locally bounded l.s.c. function $\Phi(f) : X \to \mathbb{R}$ with $|f| \le \Phi(f)$ such that $\Phi(f) \le \Phi(f')$ whenever $|f| \le |f'|$.*

The author [12] generalized 'real-valued functions' of these equivalence of Theorem 1.1 except (c), into 'maps to ordered topological vector spaces with positive interior points'. To introduce this, let us first recall some terminology.

A partially ordered real vector space $(Y, \le)$ is said to be an *ordered vector space* if the following conditions are satisfied: (i) $x \le y$ implies $x + z \le y + z$ for all $x, y, z \in Y$, (ii) $x \le y$ implies $rx \le ry$ for all $x, y \in Y$ and all $r \in \mathbb{R}$ with $r \ge 0$. The symbol $y < y'$ is used when $y \le y'$ and $y \ne y'$.

Let $(Y, \le)$ be an ordered vector space. Then, $y \in Y$ is *positive* if $\mathbf{0} \le y$, and the set $\{y \in Y : \mathbf{0} \le y\}$, called the *positive cone* of Y, is denoted by Y^+. For $y_1, y_2 \in Y$ with $y_1 \le y_2$, the subspace $(y_1 + Y^+) \cap (y_2 - Y^+)$ of Y, called an *order interval*, is denoted by $[y_1, y_2]$. A topological vector space Y is called an *ordered topological vector space* (*o.t.v.s.*, for short) if Y is an ordered vector space such that the positive cone Y^+ is closed in Y. It is known that e is an interior point of Y^+ (i.e. $e \in \mathrm{int}_Y Y^+$) if and only if $[-e, e]$ is a $\mathbf{0}$-neighborhood ([1, Lemma 2.5]). An interior point e of Y^+ is a *positive interior point* of Y if $e > \mathbf{0}$. Note that an o.t.v.s. Y is non-trivial if and only if $\mathrm{int}_Y Y^+ \subset Y^+ \setminus \{\mathbf{0}\}$, and that every o.t.v.s. with positive interior points is non-trivial. A point $x \in Y^+$ is called an *order unit* if each $y \in Y$ admits $\lambda > 0$ such that $y \le \lambda x$. The positive cone Y^+ is *normal* if each $\mathbf{0}$-neighborhood U admits a $\mathbf{0}$-neighborhood V such that $(V + Y^+) \cap (V - Y^+) \subset U$.

Let $f : X \to Y$ be a map from a topological space X into an o.t.v.s. Y. Then, f is said to be *lower semi-continuous*, *l.s.c.* for short, (resp. *upper semi-continuous*, *u.s.c.* for short), if for each $x \in X$ and each $\mathbf{0}$-neighborhood V, there exists a neighborhood O_x of x such that $f(O_x) \subset f(x) + V + Y^+$ (resp. $f(O_x) \subset f(x) + V - Y^+$) ([3], [12]). Every continuous map $f : X \to Y$ from a topological space X into an o.t.v.s. Y is l.s.c. and u.s.c., and the converse holds if Y^+ is normal.

A subset A of an o.t.v.s. Y is called *upper-bounded* [12] if for each $\mathbf{0}$-neighborhood V, there exists $n \in \mathbb{N}$ such that $A \subset nV - Y^+$. If Y has a positive interior point e, A is upper-bounded if and only if $A \subset re - Y^+$ for some $r > 0$. A map $f : X \to Y$ from a topological space X into an o.t.v.s. Y is said to be *locally*