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Edge-preserving maps of curve graphs

Jesús Hernández Hernández

Universidad Nacional Autónoma de México, Centro de Ciencias Matemáticas, UNAM, Campus Morelia, Morelia, Mich. 58190, Mexico

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ABSTRACT

Suppose S_1 and S_2 are orientable surfaces of finite topological type such that S_1 has genus at least 3 and the complexity of S_1 is an upper bound of the complexity of S_2. Let $\varphi : \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ be an edge-preserving map; then S_1 is homeomorphic to S_2, and in fact φ is induced by a homeomorphism. To prove this, we use several simplicial properties of rigid expansions, which we prove here.

0. Introduction

In this work we suppose $S_{g,n}$ is an orientable surface of finite topological type of genus $g \geq 3$ with $n \geq 0$ punctures. The *extended mapping class group of* $S_{g,n}$, denoted by $\mathrm{Mod}^*(S_{g,n})$ is the group of isotopy classes of self-homeomorphisms of $S_{g,n}$.

In 1979 (see [7]) Harvey defined the curve complex of a surface as the simplicial complex whose vertices are isotopy classes of essential curves on the surface, and whose simplices are defined by disjointness (see Section 2 for the details). We call its 1-skeleton the *curve graph of* $S_{g,n}$, which we denote by $\mathcal{C}(S_{g,n})$.

There is a natural action of $\mathrm{Mod}^*(S_{g,n})$ on the curve graph, by automorphisms. In [15] Ivanov proved that for genus at least 2 every automorphism of the curve graph is induced by a homeomorphism of $S_{g,n}$. These results were extended for most other surfaces of finite topological type by Korkmaz and Luo in [16] and [17], respectively.

Later on, Irmak (see [12], [14], [13]), Behrstock and Margalit (see [4]), and Shackleton (see [18]) generalised these results for larger classes of simplicial maps. In particular, Shackleton's result implies that any locally

E-mail address: jhdez@matmor.unam.mx.

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injective self-map of the curve graph is induced by a homeomorphism for surfaces of high-enough complexity. See Section 1 for a reminder of the terminology.

Thereafter, Aramayona and Leininger introduced in [1] the concept of a rigid set of the curve graph (described below as a rigid subgraph, in a more general setting) and constructed a finite rigid set for any orientable surface of finite topological type. Afterwards, they introduce in [2] a way of creating supersets from given sets, such that the supersets are capable of inheriting the property of being rigid (which is not trivial). This method is called the *rigid expansion* of a set in [8] and [10] due to this property.

In this article we use techniques similar to those shown in [18] along with simplicial properties of the rigid expansions to obtain the following result. Recall that the complexity of a surface is denoted by $\kappa(S_{g,n}) = 3g - 3 + n$ and is a topological invariant.

Theorem A. *Let $S_1 = S_{g_1,n_1}$ and $S_2 = S_{g_2,n_2}$ be two orientable surfaces of finite topological type such that $g_1 \geq 3$, and $\kappa(S_2) \leq \kappa(S_1)$; let also $\varphi : \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ be an edge-preserving map. Then, S_1 is homeomorphic to S_2 and φ is induced by a homeomorphism $S_1 \to S_2$.*

Note that in the context of graph theory, an edge-preserving map is a graph morphism; also, an immediate consequence of Theorem A is that if $\kappa(S_2) < \kappa(S_1)$, then there cannot exist any such φ.

Now, with Theorem A we generalise (for surfaces of genus at least 3) Shackleton's result which requires the maps to be locally injective (see [18]).

To prove Theorem A, in Section 1 we first take the simplicial interpretation of a rigid expansion and generalise it to the setting of abstract simplicial graphs:

Let Γ be a connected simplicial graph, v be a vertex of Γ and B be a set of vertices of Γ. We say v is uniquely determined by B if v is the unique vertex adjacent to every element in B. Let Y be an induced subgraph of Γ; the *first rigid expansion* of Y, denoted by Y^1, is the induced subgraph spanned by the vertices of Y and all the vertices uniquely determined by sets of vertices of Y. The n-th rigid expansion is then defined inductively: $Y^n = (Y^{n-1})^1$. We denote by Y^ω the induced subgraph spanned by the union of the vertex sets of Y^i for $i \in \mathbb{N}$. See Section 1 below for more details.

In this general setting, Theorem B below tells us in particular that given an edge-preserving simplicial map from a connected induced subgraph Y to Γ that coincides with the restriction to Y of an automorphism, the only way to extend it to Y^ω so that the extended simplicial map is at least edge-preserving, is via said automorphism of Γ.

Theorem B. *Let Γ be a connected simplicial graph, Y be a connected induced subgraph of Γ, and $\varphi : Y^\omega \to \Gamma$ be an edge-preserving map such that there exists an automorphism $\phi \in \mathrm{Aut}(\Gamma)$ with $\phi|_Y = \varphi|_Y$. Then $\varphi = \phi|_{Y^\omega}$, and any other $\psi \in \mathrm{Aut}(\Gamma)$ with $\varphi = \psi|_{Y^\omega}$ differs from ϕ by an element in $\mathrm{stab}_{pt}(Y)$.*

Similarly, we can also generalise the concept of a rigid set: we say an induced subgraph Y of Γ is *rigid* if any locally injective map $Y \to \Gamma$ is the restriction of some automorphism of Γ. With this definition and Theorem B we have the following corollary:

Corollary C. *Let Γ be a connected simplicial graph, Y be a rigid subgraph of Γ, and $\varphi : Y^\omega \to \Gamma$ be an edge-preserving map such that $\varphi|_Y$ is locally injective. Then φ is the restriction to Y^ω of an automorphism of Γ, unique up to the pointwise stabilizer of Y in $\mathrm{Aut}(\Gamma)$.*

One of the objectives of Theorem B and Corollary C, is to give a way to obtain new results on the combinatorial rigidity problem of various simplicial graphs (e.g. the pants graph, the Hatcher–Thurston graph, etc.), by one of two ways: Either by finding (suitable) subgraphs for which it can be proved that the simplicial map is induced by an automorphism of the graph, and proving that the rigid expansion of the subgraph exhaust the original graph (so we can use Theorem B); or by