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Remainders in pointfree topology

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A B S T R A C T

Remainders of subspaces are important e.g. in the realm of compactifications. Their extension to pointfree topology faces a difficulty: sublocale lattices are more complicated than their topological counterparts (complete atomic Boolean algebras). Nevertheless, the co-Heyting structure of sublocale lattices is enough to provide a counterpart to subspace remainders: the sublocale *supplements*. In this paper we give an account of their fundamental properties, emphasizing their similarities and differences with classical remainders, and provide several examples and applications to illustrate their scope. In particular, we study their behavior under image and preimage maps, as well as their preservation by pointfree continuous maps (i.e. localic maps). We then use them to characterize nearly realcompact and nearly pseudocompact frames. In addition, we introduce and study hyper-real localic maps.

1. Introduction

In general topology, by a remainder of a Tychonoff space X it is usually understood the subspace $bX \smallsetminus X$ of some compactification bX of X. Remainders of subspaces and their preservation by continuous maps play an important role in some classical results. E.g., by the Henriksen–Isbell Theorem (cf. [21]), a continuous

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map $f\colon X \to Y$ of Tychonoff spaces is *proper* (= *perfect* [19, 3.7]) if and only if any of the following equivalent conditions hold:

(R1) The Stone–Čech extension $\beta(f)\colon \beta X \to \beta Y$ of f *takes remainder to remainder*, that is,

$$\beta(f)[\beta X \smallsetminus X] \subseteq \beta Y \smallsetminus Y.$$

(R2) For every compactification κY of Y, the extension $\widetilde{f}\colon \beta X \to \kappa Y$ of f *takes remainder to remainder*, that is,

$$\widetilde{f}[\beta X \smallsetminus X] \subseteq \kappa Y \smallsetminus Y.$$

Hence, in the point-set context, remainder preserving maps are precisely the proper maps.

This provides nice categorical characterizations of proper maps since remainder preserving condition (R1) means precisely that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\beta_X"'] & Y \arrow[d, "\beta_Y"] \\ \beta X \arrow[r, "\beta(f)"] & \beta Y \end{tikzcd}$$

is a pullback diagram (i.e., f is β-cartesian [35]), while (R2) is equivalent to the fact that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\beta_X"'] & Y \arrow[d, "\kappa_Y"] \\ \beta X \arrow[r, "\widetilde{f}"] & \kappa Y \end{tikzcd}$$

is a pullback diagram. (For a broad categorical approach to properness and perfectness see [14] and [35].)

The generalization of Henriksen–Isbell Theorem to pointfree topology faces a difficulty: unlike the algebra $\mathcal{P}(X)$ of subspaces of a space X, the sublocale lattice $\mathcal{S}(L)$ of a locale (frame) L is generally not Boolean, and therefore complements (and hence the difference of two sublocales) do not necessarily exist. He and Luo [20] circumvented this by grabbing the categorical conditions rather than (R1) and (R2) to characterize proper maps of locales:

Theorem 1.1. *[20, Theorem 1] Let $f\colon L \to M$ be a localic map between completely regular locales. Then the following statements are equivalent:*

(i) *f is proper.*

(ii) *For the Stone–Čech compactification $\beta_M\colon M \to \beta M$ of M, the following diagram is a pullback square:*

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\beta_L"'] & M \arrow[d, "\beta_M"] \\ \beta L \arrow[r, "\beta(f)"] & \beta M \end{tikzcd}$$