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On the irreducibility of the space of Galois covers of curves of a given numerical type for certain split metacyclic groups ☆

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We consider the spaces $\mathfrak{M}_g(G) \subset \mathfrak{M}_g$ of curves of genus $g \geq 2$, that admit an effective action by a given finite group G and want to understand its irreducible components. We prove that if $G = C_m \rtimes C_n$ is a split metacyclic group where both factors have odd prime order, the irreducible components are determined by a numerical datum which comes from the monodromy of the induced G-coverings.

1. Introduction

The singular locus of $\mathfrak{M}_g$ (the coarse moduli space of curves of genus $g \geq 2$) consists of the loci $\mathfrak{M}_g(G)$ of the curves which admit an effective action by a finite group G. A good approach to understanding the irreducible components of $\mathfrak{M}_g(G)$ for a given finite group G is to view $\mathfrak{M}_g$ as the quotient of the Teichmüller space $\mathcal{T}_g$ by the natural action of the mapping class group Map_g:

$$\pi : \mathcal{T}_g \to \mathcal{T}_g / Map_g = \mathfrak{M}_g.$$

We have

$$\mathfrak{M}_g(G) = \bigcup_{[\rho]} \mathfrak{M}_{g,\rho}(G),$$

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where $\rho : G \hookrightarrow Map_g$ is an injective homomorphism, $\mathfrak{M}_{g,\rho}(G)$ is the image of the fixed locus of $\rho(G)$ under the natural projection π and $\rho \sim \rho'$ iff they are equivalent by the equivalence relation generated by the automorphisms of G and the conjugations by Map_g. We call this equivalence class a *topological type* (cf. [4], section 2). Each $\mathfrak{M}_{g,\rho}(G)$ is an irreducible (Zariski) closed subset of $\mathfrak{M}_g$, as proven by Catanese (cf. [4], Theorem 2.3). Primary invariants are given by the induced branched coverings $C \to C/G =: C'$, namely the genus $g' = g(C')$, the number d of branching points, and the branching orders $m_1, ..., m_d$. Every topological type moreover determines a finer numerical invariant, called the *numerical type*, given by the monodromy of the induced coverings (cf. Definition 2.2). We want to understand whether this invariant in turn determines the topological type.

Leading Question. *Does every numerical type determine a topological type?*

We prove that this holds true for split metacyclic groups with prime factors, obtaining the following result by the irreducibility of the spaces $\mathfrak{M}_{g,\rho}(G)$:

Theorem. *Let $G = C_m \rtimes C_n$ be a split metacyclic group, where $m > 3, n > 2$ are prime numbers. Then for every numerical type ν, the spaces $\mathfrak{M}_{g,\nu}(G)$ are irreducible.*

Similar results have been proven by Nielsen, Catanese and Catanese, Lönne, Perroni for cyclic groups resp. dihedral groups (cf. [9], [2] and [3], [4]). In particular, in [3], the authors proved the same result for G a dihedral group and $g' = 0$, whereas in [4] they showed that for G dihedral and $g' > 0$ the numerical type is not sufficient to distinguish the topological types, whence they introduced a new homological invariant, which turns out to be sufficient.

The practical approach in our case is as follows. By Riemann's Existence Theorem, the spaces $\mathfrak{M}_{g,\rho}(G)$ correspond to orbits of the combined action of the mapping class group and the automorphism group of G on Hurwitz vectors. Recall that a G-Hurwitz vector is a vector

$$V = (g_1, ..., g_d; a_1, b_1; ...; a_{g'}, b_{g'}) \in G^{d+2g'},$$

such that $g_i \neq 1$ for $i = 1, ..., d$, $\langle V \rangle = G$ and $\prod_{i=1}^{d} g_i \prod_{j=1}^{g'} [a_j, b_j] = 1$.

For fixed d, g', we denote by $H_{g',d}(G)$ the set of G-Hurwitz vectors with branching part $g_1, ..., g_d$ of length d and genus part $a_1, b_1, ..., a_{g'}, b_{g'}$ of length $2g'$. The goal is to prove transitivity of the action on the subset $H_{g',d,\nu}(G)$ of Hurwitz vectors of a given numerical type ν.

Accordingly, a reformulation of our theorem, and what we are practically going to show, is: *for any given numerical type, the combined action of the mapping class group and the automorphism group of G is transitive on the set of Hurwitz vectors of a given numerical type.*

The structure of the paper is as follows:

In section 2 we describe the action on Hurwitz vectors and introduce the notion of numerical type. See [11] for more details.

In section 3 we present several basic results on split metacyclic groups.

Section 4 will be as follows: First we treat the case $g' = 0$. We start by considering triples of consecutive elements, where by restriction we have an action of Br_3. We prove that there is a subgroup $H < Br_3$ which admits a linear representation $\hat{\rho} : H \to GL(3, m)$ from which we can recover the action of H on the triple. In a suitably general situation we can restrict it to a two-dimensional representation $\rho : H \to GL(2, m)$ (cf. Proposition 4.1). We show that the image of this representation ρ contains the special linear group $\mathrm{SL}(2, m)$ (cf. Proposition 4.7).

In section 5 we use that $\mathrm{SL}(2, m)$ acts transitively on non-zero vectors in $\mathbb{F}_m^2$ to establish normal forms for triples. Using these normal forms, we obtain normal forms for quadruples which we can eventually use

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