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Surplus Nielsen type numbers for periodic points on the complement

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ABSTRACT

Let $f : (X, A) \to (X, A)$ be a self-map of a pair of compact polyhedra with X connected. In 1990 Xuezhi Zhao studied Nielsen fixed point theory on the complement. He defined a surplus Nielsen number $SN(f; X - A)$ which is a good lower bound for the number of fixed points on $X - A$ for all maps in the homotopy class of f. In this paper we generalize these ideas from fixed point theory to periodic point theory, and define two Nielsen type numbers $SNP_n(f; X - A)$ and $SN\Phi_n(f; X - A)$ for periodic points on the complement. Thus we answer an open question posed by Heath and Zhao.

1. Introduction

Relative Nielsen fixed point theory studies the fixed point sets of relative maps $f : (X, A) \to (X, A)$, where (X, A) is a pair of compact polyhedra (see for example [13,15–17]). Schirmer in 1986 [13] introduced a relative Nielsen number $N(f; X, A)$ for a self-map $f : (X, A) \to (X, A)$ to take into account the fact that such a relative map f determines self-maps of both X and A, thus possibly detecting more fixed points than the ordinary Nielsen number $N(f)$. Zhao introduced relative Nielsen type numbers, $N(f; X - A)$ and $SN(f; X - A)$ in 1989 [15] and in 1990 [16], which are lower bounds for the number of fixed points on $X - A$. Relative Nielsen theory has been extended to study the number of periodic points of relative maps. In 1995 Heath, Schirmer and You combined Schirmer's ideas about relative theory introduced in [13], and the existing ideas about periodic points to estimate the number of periodic points of a map $f : (X, A) \to (X, A)$, of a pair of compact ANRs in the paper [9]. In 2000 Heath and Zhao combined periodic point theory ([12]) with the first of Zhao's complement numbers ([15]) to give estimates of the number of periodic points on

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$X - A$ of a map $f : (X, A) \to (X, A)$, of a pair of compact ANRs in the paper [11]. In the paper they introduced Nielsen type numbers $NP_n(f; X - A)$, $N\Phi_n(f; X - A)$ that give estimates for the numbers

$$M\Phi_n(f; X - A) = \min\{\#(\Phi(g^n) \bigcap (X - A)) \mid g \simeq f : (X, A) \to (X, A)\}$$

(the minimum numbers of periodic points of all periods dividing n of maps g that are homotopic to f as a map of pairs, and that lie in $X - A$), and for

$$MP_n(f; X - A) = \min\{\#(P_n(g) \bigcap (X - A)) \mid g \simeq f : (X, A) \to (X, A)\}$$

(the minimum numbers of periodic points of period exactly n of maps g that are homotopic to f as a map of pairs, and that lie in $X - A$). As pointed out in Section 7 in [11], perhaps the most interesting of the periodic numbers is the generalization of surplus fixed point theory. The main objective here then, is to introduce two new Nielsen type numbers $SNP_n(f; X - A)$ and $SN\Phi_n(f; X - A)$ (Definition 7.2 and 7.5) which will provide sharp lower bounds for the numbers $MP_n(f; X - A)$ and $M\Phi_n(f; X - A)$ respectively in most cases, and which is the generalization of surplus fixed point theory from [16]. The methods and ideas about Nielsen type numbers for periodic points shown in [3,9–12] play great parts in this paper. Our results answer an open question posed by Heath and Zhao in the final section of the paper [11]. Our work differs from that of them [11] in the way that we work on $X - A$ rather than X. We illustrate the considerations of our papers by the example for $f : (X, A) \to (X, A)$ given in the final section of the paper [11].

Example 1.1. Let $X = \mathbf{S}^1 = \{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$ be the unit circle in the complex plane, and let A be the two points $\{1, -1\}$ in X. The selfmap $f : (X, A) \to (X, A)$ is defined by $f(e^{i\theta}) = e^{2i\theta}$ if $0 \leq \theta \leq \pi$, and $f(e^{i\theta}) = e^{-2i\theta}$ if $\pi \leq \theta \leq 2\pi$. For any natural number n, f has $2^n - 1$ periodic points of periods dividing n on $X - A$, and $\Phi(f^n) \bigcap (X - A) = \{e^{i\theta_k} \mid k = 1, 2, ..., 2^n - 1;\ \theta_k = \frac{2(k+1)\pi}{2^n+1}$ for k odd, $\theta_k = \frac{2k\pi}{2^n-1}$ for k even$\}$. Please note in this situation that both the relative Nielsen type numbers for the nth iterate on the complement $N\Phi_n(f; X - A)$ and $NP_n(f; X - A)$ are zero, and therefore provide no help in determining the minimum number of periodic points on $X - A$. Let us consider surplus fixed point classes of $f^2 : (X, A) \to (X, A)$. $4\pi/3$ and $8\pi/5$ are in the same surplus fixed point class of f^2 on $X - A$, but these two points are taken to different components of $X - A$ by f. As shown in [11], the greatest (but not the only) difficulty in the generalization of surplus fixed point theory for periodic numbers, is that two periodic points in the same surplus class of f^n on the complement $X - A$ may be taken to different components of $X - A$ upon iteration by f. Our goal is to find a new periodic point classes on the complement such that each two points in the same class is taken to a single class upon iteration by f.

The paper is divided as follows. Following this introduction we give in Section 2 a brief review of the topics we use frequently in this paper. The results in the geometric side of our periodic theory on the complement are presented in Section 3 and 4. In Section 3 we define a surplus Nielsen periodic point class of relative map $f : (X, A) \to (X, A)$ on the complement, and introduce a surplus Nielsen type number $SN^n(f; X - A)$. This number is shown to be a lower bound for the number of periodic points of all periods dividing n on $X - A$ for all maps in the relative homotopy class of f. In Section 4, we introduce surplus periodic point orbits on the complement. In Section 5 and 6 we present the results in the algebraic side of our periodic theory on the complement, as well as showing the relationship between the algebraic theory and the geometric theory. In Section 5 we define algebraic Nielsen periodic point class on the complement, where we generalize the fundamental group approach introduced in [12]. In Section 6 we introduce algebraic orbits for periodic points on the complement. Finally in Section 7 we introduce two surplus Nielsen periodic point numbers $SNP_n(f; X - A)$ and $SN\Phi_n(f; X - A)$, discusses the relationship of $SNP_n(f; X - A)$ and $SN\Phi_n(f; X - A)$ to each other, and to $SN^m(f; X - A)$ for $m|n$, and to $NP_n(f; X - A)$ and $N\Phi_n(f; X - A)$.