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# On indecomposability of $\beta X$

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ABSTRACT

The following is an open problem in topology: Determine whether the Stone–Čech compactification of a widely-connected space is necessarily an indecomposable continuum. Herein we describe properties of $X$ that are necessary and sufficient in order for $\beta X$ to be indecomposable. We show that indecomposability and irreducibility are equivalent properties in compactifications of widely-connected separable metric spaces, leading to some equivalent formulations of the open problem. We also construct a widely-connected subset of Euclidean 3-space which is contained in a composant of each of its compactifications. The example answers a question of Jerzy Mioduszewski.



## 1. Introduction

This paper addresses a collection of problems relating to widely-connected sets and indecomposability of the Stone–Čech compactification.[1]

A connected topological space $W$ is *widely-connected* if every non-degenerate connected subset of $W$ is dense in $W$. A connected compact Hausdorff space (a *continuum*) is *indecomposable* if it cannot be written as the union of any two of its proper subcontinua. By Theorem 2 in [10] §48 V, the latter term can be consistently defined in the absence of compactness and/or connectedness. To wit, a topological space $X$ is *indecomposable* if every connected subset of $X$ is either dense or nowhere dense in $X$ (cf. [17,18,15] for connected spaces). Now every widely-connected space is indecomposable.

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[1] Indecomposability of the growth $\beta X \setminus X$ (otherwise known as the Stone–Čech remainder) was characterized in [7], closely following the discovery that $\beta[0,\infty)\setminus[0,\infty)$ is an indecomposable continuum – see [1] and [19] §9.12. Here we are interested in $\beta X$ as a whole.

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The three-part question below was asked by Jerzy Mioduszewski at the 2004 Spring Topology and Dynamics Conference.

**Question** *(Mioduszewski; 23 in [14]).* Let $W$ be a widely-connected space.

**(A)** Is $\beta W$ necessarily an indecomposable continuum?
**(B)** If $W$ is metrizable and separable, does $W$ necessarily have a metric compactification which is an indecomposable continuum?
**(C)** If $W$ is metrizable and separable, does $W$ necessarily have a metric compactification $\gamma W$ such that for every composant $P$ of $\gamma W$, $W \cap P$ is (i) hereditarily disconnected? (ii) finite? (iii) a singleton?

Part (A) became Problem 521 in *Open Problems in Topology II*, due to David Bellamy [4]. In Problem 520 from the same book, Bellamy conjectured a positive answer to (B).

Note that, as stated, (A) is more general than (B) and (C). Question (A) is about arbitrary Tychonoff spaces and Questions (B) and (C) are about separable metrizable spaces. We let (A′) be the version of Question (A) that assumes $W$ is separable and metrizable.

### 1.1. Notation & terminology

In Mioduszewski's question:

- $\beta W$ denotes the Stone–Čech compactification of $W$;
- $\gamma W$ is a *compactification* of $W$ if $\gamma W$ is a compact Hausdorff space in which $W$ is densely embedded;
- $P$ is a *composant* of $\gamma W$ if $P$ is the union of all proper subcontinua of $\gamma W$ that contain a given point;
- $W \cap P$ is *hereditarily disconnected* means that $|C| \leq 1$ for every connected $C \subseteq W \cap P$.

These definitions generalize in the obvious ways; see [8]. See [19] for constructions and unique properties of the Stone–Čech compactification. Basic information about composants is given in [10] §48 VI.

A subset $Q$ of a topological space $X$ is called a *quasi-component* of $X$ if there exists $q \in X$ such that

$$Q = \bigcap\{A \subseteq X : A \text{ is clopen and } q \in A\}.$$

If $|Q| = 1$ for every quasi-component $Q$ of $X$, then $X$ is *totally disconnected.*

If $p$ and $q$ are two points in a connected space $X$, then $X$ is *reducible between $p$ and $q$* if there is a closed connected $C \subsetneq X$ with $\{p, q\} \subseteq C$. Otherwise, $X$ is *irreducible between $p$ and $q$*. Observe that $W$ is widely-connected if and only if $W$ is connected and irreducible between every two of its points.

A continuum with only one composant is said to be *reducible.* A continuum is *irreducible* if it has more than one composant, that is, if there are two points between which the continuum is irreducible.

### 1.2. Summary of results & main example

Our results are divided across two sections. Results in Section 2 apply to general Tychonoff spaces, while Section 3 is reserved for the separable metrizable setting.

For Tychonoff $X$, we characterize indecomposability of $\beta X$ via an elementary property of $X$ (Theorem 4 & Corollary 5). We also prove $\beta X$ is indecomposable [resp. irreducible] if $X$ has an indecomposable [resp. irreducible] compactification (Theorems 6(i) & 6(ii)). And irreducible compactifications of indecomposable connected spaces are indecomposable (Theorem 6(iii)).