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Duality of topological modules over normed rings

Hailan Liang [a], Shu-Qi Huang [b,*]

[a] *Department of Mathematics, Fuzhou University, Fuzhou 350116, China*
[b] *School of Mathematical Sciences, Nankai University, Tianjin 300071, China*

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ABSTRACT

In this paper, we introduce the operator topology for the set of all continuous homomorphisms between two topological modules, and discuss the duality of topological modules over an admissible normed ring R (see Definition 2.1). We show that the dual functor $\mathcal{B}(-, R)$ defined on the category of locally bounded R-modules is topologically left exact. Moreover, if R is complete, and the modules are extendible, the dual functor is topologically exact.

1. Introduction

Duality is an important part of the theory of real or complex Banach spaces (see [2], [9]). More recently, many authors have studied duality in the case of locally convex spaces over non-Archimedean valued fields (see [6], [8]). In 1979, J. Flood obtain that duality holds for locally compact modules over locally compact rings, when the topology of the dual is the compact-open topology (see [5]). In 2005, S. Hernández and F.J. Trigos-Arrieta consider the duality of topological abelian groups when the topology of the dual is the precompact-open topology (see [7]).

As we know that the notion of topological modules over normed rings is a common generalization of topological vector spaces, normed rings and normed algebras. Based on this fact, our purpose in this paper is to investigate duality in the topological modules over normed rings.

Firstly, we introduce the notion of an admissible ring, which is a special normed ring and also a generalization of $\mathbb{R}, \mathbb{C}$, a non-discrete valued field and unital normed algebra. We give a useful characterization of a 0-neighborhood base in a topological module over an admissible ring.

* Corresponding author.
E-mail addresses: lghlan@163.com (H. Liang), hshq0423@126.com (S.-Q. Huang).

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Afterwards, we define the operator topology for the set of all continuous homomorphisms between two topological modules. In particular, for the topological modules over an admissible ring R, using the characterization of 0-neighborhood base, we prove that if X is a topological left R-module, then the dual module X^* with the operator topology is a topological right R-module. Moreover, we show that the dual functor $\mathcal{B}(-,R)$ is a "topologically left exact functor" (see Theorem 6.1).

Finally, we prove the open mapping theorem for the Fréchet R-modules (see Theorem 6.2), and discuss the topologically right exact property of the functor $\mathcal{B}(-,R)$ when R is complete.

In our framework, we mainly obtain some results of duality as follows:

- If X is a locally bounded module over an admissible ring, and A a submodule of X, then $(X/A)^* \simeq A^\perp$.
- If I is a left ideal of an admissible ring R, then $(R/I)^* \simeq \mathrm{ann}(I)$.
- If X is an extendible locally bounded module over a complete admissible ring, and A a submodule of X, then $A^* \simeq X^*/A^\perp$.

2. Preliminaries

Throughout this paper, we always assume that rings are all unitary.

Let R, S be two topological rings. A (R,S)-bimodule M is said to be a *topological* (R,S)-*bimodule* if M is a topological left R-module and a topological right S-module.

Remark. The definition of bimodules can be referred to [1], and the definitions of topological rings and topological left (resp. right) R-modules can be referred to [12]. If R is a topological ring, then R can be viewed as a topological (R,R)-bimodule (or topological R-bimodule for short) in a natural way.

Let M,N be two topological left (resp. right) R-modules and $\varphi : M \to N$ an R-homomorphism. We call φ a *topological isomorphism*, if φ is isomorphic and homeomorphic. In this case, we say that M,N are *topologically isomorphic*, and write $M \simeq N$ simply.

A ring R with a norm $\|\cdot\|$ defined on it is called a *normed ring*, denoted by $(R,\|\cdot\|)$ (or R for short). Furthermore, a normed ring $(R,\|\cdot\|)$ is said to be *complete* if $\|\cdot\|$ is a complete norm; a normed ring $(R,\|\cdot\|)$ is said to be *unital* if $\|1\| = 1$. See, e.g., [11,12].

It is well known that every normed ring is a topological ring.

Definition 2.1. An unital normed ring is said to be an admissible ring if there exists an invertible element λ such that $\|\lambda\| < 1$.

Remark. It is easy to see that an unital normed ring R is an admissible ring if and only if the zero element belongs to the closure of $R^\times$, where $R^\times$ is the set consisting of all invertible elements.

Examples. Every unital normed algebra is an admissible ring. For example, the normed rings of real numbers, complex numbers, quaterions, and $n \times n$ real matrices are both admissible rings. In addition, every non-discrete valued division ring is also an admissible ring. In particular, the field of p-adic numbers $\mathbb{Q}_p$ is an admissible ring.

We denote the category of abelian topological groups by $\mathcal{ATG}$.

Let R be a topological ring. The category R-$\mathcal{TM}$ (resp. $\mathcal{TM}$-R) whose objects are all topological left (resp. right) R-modules and morphisms are all continuous R-homomorphisms is called *the category of topological left (resp. right) R-modules.*

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence in $\mathcal{ATG}$ (resp. R-$\mathcal{TM}$, $\mathcal{TM}$-R). If this sequence is an exact sequence and f is a homeomorphic embedding, then $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is called a *topologically left exact sequence*.