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The functional characterizations of the Rothberger and Menger properties

Alexander V. Osipov

Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, Ural State University of Economics, Yekaterinburg, Russia

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ABSTRACT

For a Tychonoff space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. In this paper we continue to study different selectors for sequences of dense sets of $C_p(X)$ started to study in the paper [14].
A set $A \subseteq C_p(X)$ will be called 1-*dense* in $C_p(X)$, if for each $x \in X$ and an open set W in $\mathbb{R}$ there is $f \in A$ such that $f(x) \in W$.
We give the characterizations of selection principles $S_1(\mathcal{A}, \mathcal{A})$, $S_{fin}(\mathcal{A}, \mathcal{A})$ and $S_1(\mathcal{S}, \mathcal{A})$ where

- $\mathcal{A}$ — the family of 1-dense subsets of $C_p(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_p(X)$, through the selection principles of a space X. In particular, we give the functional characterizations of the Rothberger and Menger properties.

1. Introduction

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by $\mathbb{N}$. Let $\mathbb{R}$ be the real line, we put $\mathbb{I} = [0, 1] \subset \mathbb{R}$, and $\mathbb{Q}$ be the rational numbers. For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. The symbol $\mathbf{0}$ stands for the constant function to 0.

E-mail address: OAB@list.ru.

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Basic open sets of $C_p(X)$ are of the form $[x_1, ..., x_k, U_1, ..., U_k] = \{f \in C(X) : f(x_i) \in U_i,\ i = 1, ..., k\}$, where each $x_i \in X$ and each U_i is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of the point f as $\langle f, A, \epsilon\rangle$ where $\langle f, A, \epsilon\rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon\ \forall x \in A\}$, A is a finite subset of X and $\epsilon > 0$.

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space X is called sequentially separable if it has a countable sequentially dense set.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is:

- an *ω-cover* if every finite subset of X is contained in a member of $\mathcal{U}$.
- a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Note that every γ-cover contains a countably γ-cover.

For a topological space X we denote:

- $\mathcal{O}$ — the family of open covers of X;
- Γ — the family of countable open γ-covers of X;
- Ω — the family of open ω-covers of X;
- $\mathcal{D}$ — the family of dense subsets of $C_p(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_p(X)$.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Many equivalences hold among these properties, and the surviving ones appear in the Diagram (Fig. 1) (where an arrow denotes implication), to which no arrow can be added except perhaps from $U_{fin}(\Gamma, \Gamma)$ or $U_{fin}(\Gamma, \Omega)$ to $S_{fin}(\Gamma, \Omega)$ [7].

The papers [7,8,19,22,24] have initiated the simultaneous consideration of these properties in the case where $\mathcal{A}$ and $\mathcal{B}$ are important families of open covers of a topological space X.

In papers [1–5,8,9,12–14,16–21,24] (and many others) were investigated the applications of selection principles in the study of the properties of function spaces. In particular, the properties of the space $C_p(X)$ were investigated. In this paper we continue to study different selectors for sequences of dense sets of $C_p(X)$.

2. Main definitions and notation

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set.

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.