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Asymmetric norms given by symmetrisation and specialisation order [☆]

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ABSTRACT

In this paper we continue the investigations of the relationship between T_0-quasi-metric spaces and partially ordered metric spaces. Among other things, we establish an equivalence of categories between so-called maximal T_0-quasi-metric spaces and partially ordered metric spaces produced by T_0-quasi-metrics. In the linear context we give geometric interpretations of the obtained results. In particular, we also derive a representation theorem for injective asymmetrically normed spaces.

With every T_0-quasi-metric d on a set X we can associate a metric (its symmetrisation d^s) and a partial order (the specialisation order $\leq_d$ induced by d). In this way every T_0-quasi-metric space produces a partially ordered metric space $(X, d^s, \leq_d)$. It is natural to ask when the converse will be true, that is, given a partially ordered metric space $(X, m, \leq)$, when will there be a T_0-quasi-metric d that produces $(X, m, \leq_d)$ in the sense that $m = d^s$ and the partial orders $\leq$ and $\leq_d$ on X coincide? This question was investigated in detail in [7] and [8], where it was shown that in the presence of additional algebraic structure on X a satisfactory characterisation is possible, and that order-convexity plays an important role in this regard.

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In this paper we continue the investigation of the relationship between T_0-quasi-metric spaces and partially ordered metric spaces. We show that it is possible to set up a Galois connection between these two classes of spaces. The notions of a *maximal* T_0-quasi-metric space and a *produced* partially ordered metric space come up naturally when considering the closure and kernel operators associated with this connection, and restricted to these spaces the Galois connection becomes a one-to-one correspondence. We further show that there is a close relationship between non-expansive maps between T_0-quasi-metric spaces and non-expansive and increasing maps between partially ordered metric spaces. This allows us to set up a functorial relationship between appropriate categories which becomes an equivalence of categories when we restrict to maximal T_0-quasi-metric spaces and produced partially ordered metric spaces.

We look at the linear context, where the underlying space X is a real vector space, in some detail. Here the T_0-quasi-metric is induced by an asymmetric norm, and the metric and partial order of the partially ordered metric space is induced by a norm and a cone respectively. The Galois connection can be re-interpreted as a connection between appropriate subsets of X, and this enables us to give a more geometric interpretation of earlier results in this context.

In the final section we show that the equivalence of categories established earlier allows us to make use of a known representation theorem for injective partially ordered normed spaces to derive a representation for injective asymmetrically normed spaces, thus answering a question left open in [5].

1. Preliminaries

A *quasi-pseudo-metric* on a set X is a function $d : X \times X \to [0, \infty)$ such that $d(x, x) = 0$ for all $x \in X$ and $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$. If, in addition, $d(x, y) = 0 = d(y, x)$ implies $x = y$, d is a T_0-*quasi-metric*; if also $d(x, y) = d(y, x)$ for all $x, y \in X$, d is a *metric*. The *symmetrisation* of the T_0-quasi-metric d is the metric d^s defined, for $x, y \in X$, by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$. For a quasi-pseudo-metric d the associated *specialisation pre-order* on X is defined by $x \le_d y$ if and only if $d(x, y) = 0$. If d is a T_0-quasi-metric, $\le_d$ is a partial order.

A function $p : X \to [0, \infty)$ on a real vector space X will be called an *asymmetric seminorm* on X if for all $x, y \in X$ and $\lambda \in [0, \infty)$, $p(\lambda x) = \lambda p(x)$ and $p(x + y) \le p(x) + p(y)$. If $p(x) = p(-x) = 0$ implies $x = 0$, p will be called an *asymmetric norm*.

If p is an asymmetric norm on X, the function $p^t : X \to [0, \infty)$ defined by $p^t(x) = p(-x)$ for $x \in X$ is also an asymmetric norm, the asymmetric norm conjugate to p.

The symmetrisation of the asymmetric norm p is the function $p^s : X \to [0, \infty)$ given by

$$p^s(x) = \max\{p(x), p(-x)\}, \quad x \in X$$

and this is easily seen to be a norm on X.

An asymmetric norm p induces a T_0-quasi-metric d_p on X defined by $d_p(x, y) = p(x - y)$ for all $x, y \in X$. The associated partial order is defined by $x \le_p y$ if and only if $p(x - y) = 0$.

A non-empty subset A of the real vector space X is *absorbent* if for every $x \in X$, there is a $\lambda > 0$ such that $x \in \lambda A$, and *balanced* if $\lambda x \in A$ for every $x \in A$ and for all $|\lambda| \le 1$. A set that is both convex and balanced is called *absolutely convex*. It is clear that every absorbent set contains 0. If A is a convex set containing 0, then $0 < \lambda \le \mu$ implies $\lambda A \subseteq \mu A$. If A is both convex and absorbent (but not necessarily balanced), then for every $x \in X$, there is a $\lambda > 0$ such that $x \in \mu A$ for every $\mu \ge \lambda$.

If A is an absorbent set in X, the *gauge* p_A of A is defined by

$$p_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}, \quad x \in X.$$