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Genera of two-bridge knots and epimorphisms of their knot groups

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ABSTRACT

Let K, K' be two-bridge knots of genus k, k' respectively. We show the necessary and sufficient condition of k in terms of k' that there exists an epimorphism from the knot group of K onto that of K'.

1. Introduction

Let K be a knot in S^3 and $G(K)$ the knot group, that is, the fundamental group of the complement of K in S^3. We denote by $g(K)$ the genus of K. Recently, many papers have investigated epimorphisms between knot groups. In particular, Simon's conjecture in [10], which states that every knot group maps onto at most finitely many knot groups, was settled affirmatively in [2]. In the same Kirby's problem list [10], Simon also proposed another conjecture. Namely, if there exists an epimorphism from $G(K)$ onto $G(K')$, then is $g(K)$ greater than or equal to $g(K')$? This problem is also mentioned in [11]. It is known that if there exists an epimorphism from $G(K)$ onto $G(K')$, then the Alexander polynomial of K is divisible by that of K'. Moreover, Crowell [8] showed that the genus of an alternating knot is equal to a half of the degree of the Alexander polynomial. Then the above conjecture is true for alternating knots, especially two-bridge knots.

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In this paper, we give a more explicit condition on genera of two-bridge knots K and K' such that there exists an epimorphism between their knot groups. As a corollary, we show that if there exists an epimorphism from $G(K)$ onto $G(K')$, where K is a two-bridge knot, then $g(K) \geq 3g(K') - 1$.

A knot is called *minimal* if its knot group admits epimorphisms onto the knot groups of only the trivial knot and itself. Many types of minimal knots have already been shown in [5], [12], [14], [16], [17], [18], and [20]. By using the main theorem of this paper, we obtain several types of minimal knots. For example, a two-bridge knot of genus 2 is minimal if and only if it is not the two-bridge knot $C[2a, 4b, 4a, 2b]$ in Conway's notation for any non-zero integers a, b.

2. The Ohtsuki–Riley–Sakuma construction

In this section, we review some known facts about two-bridge knots, see [6] and [15] for example. Especially, we recall the Ohtsuki–Riley–Sakuma construction of epimorphisms between two-bridge knot groups.

It is known that a two-bridge knot corresponds to a rational number. Then $K(r)$ stands for the two-bridge knot corresponding to a rational number r. Furthermore, a rational number r can be expressed as a continued fraction

$$r = [a_1, a_2, \dots, a_{m-1}, a_m] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_{m-1} + \cfrac{1}{a_m}}}}},$$

and $a_1 > 0$. We define the *length* of the continued fraction to be

$$\ell([a_1, a_2, \dots, a_{m-1}, a_m]) = m.$$

Note that the length depends on the choice of continued fractions. For example, we can delete zeros in a continued fraction by using

$$\begin{aligned} &[a_1, a_2, \dots, a_{i-2}, a_{i-1}, 0, a_{i+1}, a_{i+2}, \dots, a_m] \\ = &[a_1, a_2, \dots, a_{i-2}, a_{i-1} + a_{i+1}, a_{i+2}, \dots, a_m]. \end{aligned}$$

Then, we can reduce the length by 2, if the continued fraction contains a 0.

Theorem 2.1 *(Ohtsuki–Riley–Sakuma [19], Agol [1], Aimi–Lee–Sakuma [3]). Let $K(r), K(\tilde{r})$ be two-bridge knots, where $r = [a_1, a_2, \dots, a_m]$. There exists an epimorphism $\varphi : G(K(\tilde{r})) \to G(K(r))$ if and only if $\tilde{r}$ can be written as*

$$\tilde{r} = [\varepsilon_1 \mathbf{a}, 2c_1, \varepsilon_2 \mathbf{a}^{-1}, 2c_2, \varepsilon_3 \mathbf{a}, 2c_3, \varepsilon_4 \mathbf{a}^{-1}, 2c_4, \dots, \varepsilon_{2n} \mathbf{a}^{-1}, 2c_{2n}, \varepsilon_{2n+1} \mathbf{a}],$$

where $\mathbf{a} = (a_1, a_2, \dots, a_m), \mathbf{a}^{-1} = (a_m, a_{m-1}, \dots, a_1)$, $\varepsilon_i = \pm 1$ ($\varepsilon_1 = 1$), $c_i \in \mathbb{Z}$, and $n \in \mathbb{N}$.

Remark 2.2. We can exclude the case where $c_i = 0$ and $\varepsilon_i \cdot \varepsilon_{i+1} = -1$ without loss of generality, since we can reduce the length of $\tilde{r}$ in this case (see [20, Remark after Theorem 3.1] for details).

Remark 2.3. The "if" part of Theorem 2.1 is given by Theorem 1.1 of [19]. Conversely, the statement for "only if" part is shown by a similar argument to Proof of Theorem 1.4 of [13]. Lee and Sakuma considered only genus-one two-bridge knots in Theorem 1.4 in [13]. In order to generalize this argument, we need