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Non-meagre subgroups of reals disjoint with meagre sets

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ABSTRACT

Let $(X, +)$ denote $(\mathbb{R}, +)$ or $(2^\omega, +_2)$. We prove that for any meagre set $F \subseteq X$ there exists a subgroup $G \leq X$ without the Baire property, disjoint with some translation of F. We point out several consequences of this fact and indicate why analogous result for the measure cannot be established in ZFC. We extend proof techniques from [1].

1. Historical background

For sets $A, B \subseteq \mathbb{R}$ we define the algebraic sum $A + B = \{a + b | \, a \in A, \, b \in B\}$. Study of algebraic sums of this kind has been around for almost a century. The first result in this topic seems to be due to Sierpiński, who proved in 1920 that there exists two sets of measure zero whose sum is non-measurable [4]. Rubel [12] showed that these two sets can be chosen to be equal. This result was later generalized in many directions. For example, related results for other σ-ideals were obtained by Kharazishvili [10] and by Cichoń and Jasiński [5]. In another direction, Ciesielski, Fejzić and Freiling [6] proved among others, that for every set $C \subseteq \mathbb{R}$, there exists a set $A \subset C$ such that $\lambda_*(A + A) = 0$ and $\lambda^*(A + A) = \lambda^*(C + C)$, where λ_* and λ^* denote the inner and the outer Lebesgue measure respectively (but for simpler proof see the work by Marcin Kysiak [13]). It is also worth to mention the famous Erdös–Kunen–Mauldin theorem [9].

It is easy to see that the sum of compact (open) sets is compact (open), the sum of F_σ sets is F_σ, but for higher Borel classes this is not the case [5]. Even the sum of a compact set with G_δ doesn't have to be a Borel set, and this was shown by Sodnomow in 1954 [11] and independently by Erdös and Stone in 1970 [8].

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The study of algebraic sums of subsets of real line is closely related to the study of additive subgroups of $(\mathbb{R},+)$. Erdös proved, that under CH there exists a non-meagre, null additive subgroup of reals, as well as a non-measurable, meagre additive subgroup of reals [7]. The same can be proved under MA, but somehow surprisingly, while non-meagre subgroups of measure zero always exists, some additional set-theoretic assumption turns out to be necessary to prove the existence of a subgroup which is non-measurable and meagre. This was proved recently by Rosłanowski and Shelah [1].

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2. Preliminaries

In private communication Sergei Akbarov posed the following problem, sometimes referred to as the Akbarov Problem.

Problem 1. Let $A \subseteq \mathbb{R}$ be a nonempty null set. Does there exist a set $B \subseteq \mathbb{R}$ with the property that $A+B$ is Lebesgue non-measurable?

One of the natural ways to approaching such problem is to try to find a non-measurable dense subgroup $G \leq \mathbb{R}$ disjoint with some translation of A. Indeed, assume we have a non-measurable dense subgroup $G \leq \mathbb{R}$, and $(A+v) \cap G = \emptyset$ holds. Then also

$$(A+v) \cap (G-G) = \emptyset,$$
$$(A+v+G) \cap G = \emptyset.$$

It is well-known (see for example [3], Thm. 7.36) that every dense subgroup of $\mathbb{R}$ is either null or has full outer measure. Both G and $A+v+G$ have full outer measure, hence both have inner measure zero, and so are non-measurable.

This approach, however sufficient to solve the problem with certain additional assumptions like Martin's Axiom, won't work in ZFC alone. In 2016 Andrzej Rosłanowski and Saharon Shelah proved the following theorem [1].

Theorem 1. *It is relatively consistent with ZFC that any meagre subgroup of reals is null.*

Indeed, consider a dense G_δ null subset of reals. Any subgroup disjoint with its translation must be meagre. So, consistently, also null.

In the case of Baire category however, situation is different. The following is the main result of this paper.

Main Theorem. *Let $X = (\mathbb{R},+)$ or $X = (2^\omega, +_2)$. For any meagre set $F \subseteq X$, there exists $x \in X$, and a dense subgroup $H \leq X$ without the Baire property such that $(F+x) \cap H = \emptyset$.*

From this follows the affirmative answer to the category version of Problem 1.

Corollary 1. *For any meagre set $A \subseteq X$, there exist a set $B \subseteq X$ such that $A+B$ doesn't have the Baire property.*

Proof. Just take as B a dense subgroup without the Baire property, which is disjoint with a translation of A. □