Topology and its Applications 241 (2018) 78–81

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Seifert surgery on knots via Reidemeister torsion and Casson–Walker–Lescop invariant III

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ARTICLE INFO

Article history:
Received 30 October 2017
Received in revised form 16 March 2018
Accepted 22 March 2018
Available online 29 March 2018

MSC:
11R04
11R27
57M25
57M27

Keywords:
Reidemeister torsion
Casson–Walker–Lescop invariant
Seifert fibered space

ABSTRACT

For a knot K in a homology 3-sphere Σ, let M be the result of $2/q$-surgery on K, and let X be the universal abelian covering of M. Our first theorem is that if the first homology of X is finite cyclic and M is a Seifert fibered space with $N \geq 3$ singular fibers, then $N \geq 4$ if and only if the first homology of the universal abelian covering of X is infinite. Our second theorem is that under an appropriate assumption on the Alexander polynomial of K, if M is a Seifert fibered space, then $q = \pm 1$ (i.e. integral surgery).

1. Introduction

It is conjectured that Seifert surgeries on non-trivial knots are integral (except some cases), and that the resultant Seifert fibered spaces have three singular fibers (for example, see [2]). We [5,6] have studied the $2/q$-Seifert surgeries. One reason why the coefficients are $2/q$ is that we started from the following shown in [3]:

Theorem 1.1. ([3, Theorem 1.4]) *Let K be a knot in a homology 3-sphere Σ such that the Alexander polynomial of K is $t^2 - 3t + 1$. The only surgeries on K that may produce a Seifert fibered space with*

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https://doi.org/10.1016/j.topol.2018.03.034
0166-8641/© 2018 Published by Elsevier B.V.

base S^2 and with $H_1 \neq \{0\}, \mathbb{Z}$ have coefficients $2/q$ and $3/q$, and produce Seifert fibered space with three singular fibers. Moreover (1) if the coefficient is $2/q$, then the set of multiplicities is $\{2\alpha, 2\beta, 5\}$ where $\gcd(\alpha, \beta) = \gcd(\alpha, 5) = \gcd(\beta, 5) = 1$*, and (2) if the coefficient is $3/q$, then the set of multiplicities is $\{3\alpha, 3\beta, 4\}$ where* $\gcd(\alpha, \beta) = \gcd(\alpha, 2) = \gcd(\beta, 2) = 1$*.*

In this paper, we continue to study the $2/q$-Seifert surgeries, where we replace the condition on the Alexander polynomial with a condition on the first homology of the double branched covering. We note that the results below are related to the conjectures mentioned above.

(1) Let Σ be a homology 3-sphere, and let K be a knot in Σ. Then $\Delta_K(t)$ denotes the Alexander polynomial of K, $a_2(K)$ denotes the Conway's a_2-term of K, and $\Sigma(K; p/r)$ denotes the result of p/r-surgery on K.

(2) The first author [4] introduced the norm of polynomials and homology lens spaces: Let ζ_d be a primitive d-th root of unity. For an element α of $\mathbb{Q}(\zeta_d)$, $N_d(\alpha)$ denotes the norm of α associated to the algebraic extension $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$. Let $f(t)$ be a Laurent polynomial over $\mathbb{Z}$. We define $|f(t)|_d$ by

$$|f(t)|_d = |N_d(f(\zeta_d))| = \left| \prod_{i \in (\mathbb{Z}/d\mathbb{Z})^\times} f(\zeta_d^i) \right|.$$

Let X be a homology lens space with $H_1(X) \cong \mathbb{Z}/p\mathbb{Z}$. Then there exists a knot K in a homology 3-sphere Σ such that $X = \Sigma(K; p/r)$ ([1, Lemma 2.1]). We define $|X|_d$ and $\|X\|_d$ by

$$|X|_d = |\Delta_K(t)|_d \quad \text{and} \quad \|X\|_d = \prod_{d' | d, d' \geq 1} |X|_{d'},$$

where d is a divisor of p. Then both $|X|_d$ and $\|X\|_d$ are topological invariants of X (refer to [4] for details).

(3) Let X be a closed oriented 3-manifold. Then $\lambda(X)$ denotes the Lescop invariant of X ([7]). Note that $\lambda\left(S^3\right) = 0$.

2. Results

Let K be a knot in a homology 3-sphere Σ. For an odd integer q, let M be the result of $2/q$-surgery on K: $M = \Sigma(K; 2/q)$. Let $\pi : X \to M$ be the universal abelian covering of M (i.e. the covering associated to $\mathrm{Ker}(\pi_1(M) \to H_1(M))$). Since $H_1(M) \cong \mathbb{Z}/2\mathbb{Z}$, π is the 2-fold unbranched covering.

Let Σ_2 be the double branched covering space of Σ branched along K, and $\overline{K}$ the lifted knot of K in Σ_2. Since $\overline{K}$ is null-homologous in Σ_2, and X is the result of $1/q$-surgery on $\overline{K}$, we have $H_1(X) \cong H_1(\Sigma_2)$.

$$\begin{array}{ccccccl} \overline{K} & \subset & \Sigma_2 & \rightsquigarrow & X & = & \Sigma_2(\overline{K}; 1/q) \\ & & \downarrow & & \downarrow & & \\ K & \subset & \Sigma & \rightsquigarrow & M & = & \Sigma(K; 2/q) \end{array}$$

Assume that $H_1(\Sigma_2) \cong \mathbb{Z}/m\mathbb{Z}$ with $m \geq 3$. Then $H_1(X) \cong \mathbb{Z}/m\mathbb{Z}$. Hence $|X|_d$ is defined for each divisor d of m, and so is $\|X\|_d$.

We then have the following.

Theorem 2.1. *Let K be a knot in a homology 3-sphere Σ. With the notation above, we assume that $H_1(\Sigma_2) \cong \mathbb{Z}/m\mathbb{Z}$ with $m \geq 3$, and $M = \Sigma(K; 2/q)$ is a Seifert fibered space with N singular fibers where $N \geq 3$. Then the following hold.*