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Uniform boundedness in function spaces

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A B S T R A C T

We investigate several boundedness properties of function spaces considered as uniform spaces.

1. Introduction

We begin with some basic information about uniform and function spaces. Our topological notation and terminology are standard (see [5]). By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of natural and real numbers, respectively.

1.1. Uniform spaces

Let X be a nonempty set. A family $\mathbb{U}$ of subsets of $X \times X$ satisfying the conditions

(U1) each $U \in \mathbb{U}$ contains the diagonal $\Delta_X = \{(x,x) : x \in X\}$ of X;
(U2) if $U, V \in \mathbb{U}$, then $U \cap V \in \mathbb{U}$;

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(U3) if $U \in \mathbb{U}$ and $V \supset U$, then $V \in \mathbb{U}$;
(U4) for each $U \in \mathbb{U}$ there is $V \in \mathbb{U}$ with $V \circ V := \{(x,y) \in X \times X : \exists z \in V \text{ such that } (x,z) \in V, (z,y) \in V\} \subset U$;
(U5) for each $U \in \mathbb{U}$, $U^{-1} := \{(x,y) \in X \times X : (y,x) \in U\} \in \mathbb{U}$

is called a *uniformity* on X.

Elements of the uniformity $\mathbb{U}$ are called *entourages*. For any entourage $U \in \mathbb{U}$, a point $x \in X$ and a subset A of X one defines the set

$$U[x] := \{y \in X : (x,y) \in U\}$$

called the *U-ball with the center x*, and the set

$$U[A] := \bigcup_{a \in A} U[a]$$

called the *U-neighborhood* of A.

In [11], several boundedness properties of uniform spaces were introduced and studied. We recall the definitions of those properties.

Definition 1.1. A uniform space $(X, \mathbb{U})$ is called:

(1) *totally bounded* (resp. *ω-bounded*) if for each $U \in \mathbb{U}$ there is a finite (resp. countable) set $A \subset X$ such that $X = U[A]$. X is *σ-totally bounded* if it is a union of countably many totally bounded subspaces;
(2) *Menger bounded* (or M-*bounded* for short) if for each sequence $(U_n : n \in \mathbb{N})$ of entourages there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[F_n]$ [11,12];
(3) *Hurewicz bounded* (or H-*bounded* for short) if for each sequence $(U_n : n \in \mathbb{N})$ of entourages there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many $U_n[F_n]$ [11,12];
(4) *Rothberger bounded* (or R-*bounded* for short) if for each sequence $(U_n : n \in \mathbb{N})$ of entourages there is a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[x_n]$ [11,12].

To each of the above boundedness properties one can correspond a game on $(X, \mathbb{U})$. For example, the game corresponded to M-boundedness is the following. Players ONE and TWO play a round for each $n \in \mathbb{N}$. In the n-th round ONE chooses an element $U_n \in \mathbb{U}$, and TWO responds by choosing a finite set $A_n \subset X$. TWO wins a play

$$U_1, A_1; U_2, A_2; \cdots ; U_n, A_n; \cdots$$

if $X = \bigcup_{n\in\mathbb{N}} U_n[A_n]$; otherwise ONE wins.

A uniform space $(X, \mathbb{U})$ is said to be *strictly* M-*bounded* if TWO has a winning strategy in the above game ([11,12]).

In a similar way we define the games associated to H-boundedness and R-boundedness, and *strictly* H-*bounded* and *strictly R-bounded* uniform space.

1.2. Function spaces

Let X be a Tychonoff space, (Y, d) be a metric space and $C(X, Y)$ be the set of continuous functions from X to Y. In case $Y = \mathbb{R}$ we write $C(X)$ instead of $C(X, \mathbb{R})$. If $y \in (Y, d)$ and $\lambda > 0$, we put $S(y, \lambda) = \{z \in Y : d(y,z) < \lambda\}$ and $B(y, \lambda) = \{z \in Y : d(y,z) \leq \lambda\}$.