Topology and its Applications 241 (2018) 252–262

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On a Van Kampen theorem for Hawaiian groups

Ameneh Babaee, Behrooz Mashayekhy *, Hanieh Mirebrahimi, Hamid Torabi

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

A R T I C L E I N F O

Article history:
Received 13 July 2017
Received in revised form 17 March 2018
Accepted 8 April 2018
Available online 10 April 2018

MSC:
55Q05
55Q20
55P65
55Q52

Keywords:
Hawaiian group
Hawaiian earring
Van Kampen theorem
Griffiths space

A B S T R A C T

The paper is devoted to study the nth Hawaiian group $\mathcal{H}_n$, $n \geq 1$, of the wedge sum of two spaces $(X, x_*) = (X_1, x_1) \vee (X_2, x_2)$. We are going to give some versions of the van Kampen theorem for Hawaiian groups of the wedge sum of spaces. First, among some results on Hawaiian groups of semilocally strongly contractible spaces, we present a structure for the nth Hawaiian group of the wedge sum of CW-complexes. Second, we give more informative structures for the nth Hawaiian group of the wedge sum X, when X is semilocally n-simply connected at x_*. Finally, as a consequence, we study Hawaiian groups of Griffiths spaces for all dimensions $n \geq 1$ to give some information about their structure at any points.

1. Introduction and motivation

In 2006, U.H. Karimov and D. Repovš [6] defined the nth Hawaiian group as a covariant functor $\mathcal{H}_n : hTop_* \to Groups$ from the pointed homotopy category, $hTop_*$, to the category of all groups, $Groups$, for $n \geq 1$. For any pointed space (X, x_0), the nth Hawaiian group $\mathcal{H}_n(X, x_0)$ was defined to be the set of all pointed homotopy classes $[f]$, where $f : (\mathbb{HE}^n, \theta) \to (X, x_0)$ is a pointed map. Here, $\mathbb{HE}^n$ denotes the n-dimensional Hawaiian earring, the union of n-spheres $\mathbb{S}^n_k$ in $\mathbb{R}^{n+1}$ with centre $(1/k, 0, \ldots, 0)$ and radius $1/k$, and the origin θ is considered as the base point. The operation of the nth Hawaiian group arises component-wisely from the operation of the nth homotopy group. Thus the following map

$$\varphi : \mathcal{H}_n(X, x_0) \to \prod_{\mathbb{N}} \pi_n(X, x_0), \tag{I}$$

* Corresponding author.
E-mail addresses: Am.Babaee@um.ac.ir (A. Babaee), bmashf@um.ac.ir (B. Mashayekhy), h_mirebrahimi@um.ac.ir (H. Mirebrahimi), h.torabi@ferdowsi.um.ac.ir (H. Torabi).

https://doi.org/10.1016/j.topol.2018.04.005
0166-8641/

defined by $\varphi([f]) = ([f|_{\mathbb{S}^n_1}], [f|_{\mathbb{S}^n_2}], ...)$ is a homomorphism, for all $n \in \mathbb{N}$. Note that in general, the homomorphism φ is not injective nor surjective. Let $C(\mathbb{HE}^1)$ be the cone over 1-dimensional Hawaiian earring. Since $C(\mathbb{HE}^1)$ is simply connected and $\mathcal{H}_1(C(\mathbb{HE}^1), \theta)$ is not trivial [6], φ can not be injective. Moreover, for every locally n-simply connected first countable space X, $im(\varphi) = \prod^W_{\mathbb{N}} \pi_n(X, x_0)$ [6, Theorem 1] which is a proper subgroup of $\prod_{\mathbb{N}} \pi_n(X, x_0)$ whenever $\pi_n(X, x_0)$ is nontrivial. Here, we consider the weak direct product $\prod^W_{\mathbb{N}} \pi_n(X, x_0)$ as a subgroup of $\mathcal{H}_n(X, x_0)$, because it is proved that for every pointed space (X, x_0), $\prod^W_{\mathbb{N}} \pi_n(X, x_0)$ can be embedded as a normal subgroup in $\mathcal{H}_n(X, x_0)$ (see [1, Lemma 2.4] and [1, Theorem 2.13]).

Although the nth Hawaiian group functor is a pointed homotopy invariant functor on the category of all pointed topological spaces, it is not freely homotopy invariant. Because unlike other homotopy invariant functors, Hawaiian groups of contractible spaces are not necessarily trivial. Karimov and Repovš [6] gave a contractible space, the cone over $\mathbb{HE}^1$, with nontrivial 1st Hawaiian group at some points (consisting of the points at which $C\mathbb{HE}^1$ is not locally 1-simply connected), but with trivial homotopy, homology and cohomology groups. More precisely, it can be shown that $\mathcal{H}_1(C(\mathbb{HE}^1), \theta)$ is uncountable, using [6, Theorem 2]: "*if X has a countable local basis at x_0, then countability of nth Hawaiian group $\mathcal{H}_n(X, x_0)$ implies locally n-simply connectedness of X at x_0.*" Furthermore, a converse of the above statement can be found in [1, Corollaries 2.16 and 2.17]: "*let X have a countable local basis at x_0. Then $\mathcal{H}_n(CX, \tilde{x})$ is trivial if and only if X is locally n-simply connected at x_0 and it is uncountable otherwise*". In addition, unlike homotopy groups, Hawaiian groups of pointed space (X, x_0) depend on the behaviour of X at x_0, and then their structures depend on the choice of base point. In this regard, there exist some examples of path connected spaces with non-isomorphic Hawaiian groups at different points, such as the n-dimensional Hawaiian earring, where $n \geq 2$ (see [1, Corollary 2.11]).

Despite the above different behaviours between Hawaiian groups and homotopy groups, they have some similar behaviours. For instance, it was proved that similar to the nth homotopy group, the nth Hawaiian group of any pointed space is abelian, for all $n \geq 2$ [1, Theorem 2.3]. Also, the Hawaiian groups preserve products in the category $hTop_*$ [1, theorem 2.12].

In this paper, we investigate the Hawaiian groups of the coproduct in the category $hTop_*$ which is the wedge sum of a given family of pointed spaces. In fact, we are going to give some versions of the van Kampen theorem for Hawaiian groups of the wedge sum of spaces. In Section 2, among some results on Hawaiian groups of semilocally strongly contractible spaces, we intend to present a structure for the nth Hawaiian group of the wedge sum of CW-complexes. A given space X is semilocally strongly contractible at x_0 if the inclusion $i : U \hookrightarrow X$ is nullhomotopic in X relative to $\{x_0\}$, for some open neighbourhood U of x_0 (see [4]).

In Section 3, we present the nth Hawaiian group of the wedge sum $(X, x_*) = (X_1, x_1) \vee (X_2, x_2)$ as the direct product of two its subgroups that are more perceptible, when X is semilocally n-simply connected at x_*. Also, we prove that the Hawaiian group of a pointed space equals the Hawaiian group of every neighbourhood of the base point if all n-loops are small. An n-loop $\alpha : (\mathbb{S}^n, 1) \to (X, x)$ is called small, if for each neighbourhood U of x, α has a homotopic representative in U (see [7,10]).

In Section 4, we investigate the Hawaiian groups of the nth Griffiths space, introduced in [4, Corollary 1.4], by generalizing the well-known Griffiths space, as the wedge sum of two copies of the cone over the n-dimensional Hawaiian earring. For the sake of clarity, we call the well known Griffiths space as the 1st Griffiths space. Then, using results of Sections 2 and 3, we intend to give some information about the structure of Hawaiian groups of Griffiths spaces at any points.

In this paper, all homotopies are relative to the base point, unless stated otherwise.