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Topologies associated with the one point compactifications of Khalimsky topological spaces

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ABSTRACT

In this paper, after discussing the one point compactification of the Khalimsky line (*resp.* the Khalimsky plane), denoted by $(\mathbf{Z}^*, \kappa^*)$ (*resp.* $((\mathbf{Z}^2)^*, (\kappa^2)^*)$), we study various properties of these compactifications associated with the semi-$T_{\frac{1}{2}}$ axiom, a non-Alexandroff structure, a non-cut-point space and so forth. We also investigate dense subsets and nowhere dense subsets of $(\mathbf{Z}^*, \kappa^*)$ and $((\mathbf{Z}^2)^*, (\kappa^2)^*)$. Finally, motivated by a particular point topology and an excluded point topology, we develop two kinds of new topologies as quotient topological spaces of $(\mathbf{Z}^*, \kappa^*)$ called an excluded two points topology and a cofinite particular point topology.

1. Introduction

Among many kinds of compactifications, the paper concerns one point compactifications of Khalimsky topological spaces such as the Khalimsky line and the Khalimsky plane [15]. Let $(\mathbf{Z}, \kappa)$ (*resp.* $(\mathbf{Z}^2, \kappa^2)$) be the Khalimsky line (*resp.* the Khalimsky plane). It is well known that each of $(\mathbf{Z}, \kappa)$ and $(\mathbf{Z}^2, \kappa^2)$

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is neither a compact nor a Hausdorff space but a locally compact space. Hence we adapt Alexandroff's one point compactification [1] to both $(\mathbf{Z}, \kappa)$ and $(\mathbf{Z}^2, \kappa^2)$. Hereafter, we denote by $(\mathbf{Z}^*, \kappa^*)$ (*resp.* $((\mathbf{Z}^2)^*, (\kappa^2)^*)$ by the one point compactification of $(\mathbf{Z}, \kappa)$ (*resp.* $(\mathbf{Z}^2, \kappa^2)$). Then we study various properties of $(\mathbf{Z}^*, \kappa^*)$ and $((\mathbf{Z}^2)^*, (\kappa^2)^*)$ and further, quotient topological structures of $(\mathbf{Z}^*, \kappa^*)$ and Khalimsky line segments.

Based on these studies, we address the following:

• Although each T_0 Alexandroff topological space is a semi-$T_{\frac{1}{2}}$ space [5], we prove that the converse does not hold.

• Development of an example showing that the closure of a compact space need not be compact.

• We prove that $(\mathbf{Z}^*, \kappa^*)$ is not an Alexandroff space.

• We prove that (X^*, κ^*_X) and (Y^*, κ^*_Y) may be homeomorphic for the non-homeomorphic Khalimsky line segments (X, κ_X) and (Y, κ_Y) with the same odd cardinality.

• Development of new topologies as quotient topological spaces of $(\mathbf{Z}^*, \kappa^*)$ induced by two canonical maps.

Since we will often use "Khalimsky" in this paper, hereafter we will use the terminology "K-" instead of "Khalimsky", if there is no danger of ambiguity.

This paper is organized as follows. Section 2 provides some basic notions on K-topology. Section 3 studies some properties of $(\mathbf{Z}^*, \kappa^*)$ and $((\mathbf{Z}^2)^*, (\kappa^2)^*))$ such as the semi-$T_{\frac{1}{2}}$ axiom, non-Alexandroff structure, non-cut-point space and so forth. Section 4 investigates dense subsets or nowhere dense subsets of $(\mathbf{Z}^*, \kappa^*)$ and $((\mathbf{Z}^2)^*, (\kappa^2)^*)$. Section 5 considers types of unstudied spaces that arise as quotients of $(\mathbf{Z}^*, \kappa^*)$ called an excluded two points topology and a cofinite particular point topology in the present paper. Section 6 concludes the paper with summary and a further work.

2. Preliminaries

Let us investigate various properties of Khalimsky topological spaces (defined later in this paragraph) associated with separation axioms and the Alexandroff topological structure. Let us recall basic notions related to this work. We say that a topological space (X, T) is an Alexandroff space if every point $x \in X$ has a smallest (or minimal) open neighborhood in (X, T) [2]. Motivated by the Alexandroff topological structure [1–3], the Khalimsky nD space, denoted by $(\mathbf{Z}^n, \kappa^n)$, was established and the study of its properties includes the papers [8,12,14–17,22,24]. Let us now recall basic notions of the Khalimsky nD space. The *Khalimsky line topology* κ on $\mathbf{Z}$, denoted by $(\mathbf{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} \mid n \in \mathbf{Z}\}$ as a subbase [2] (see also [16]), where for $a, b \in \mathbf{Z}$, $[a, b]_{\mathbf{Z}} := \{X \in \mathbf{Z} \mid a \le x \le b\}$. In the present paper we call $([a, b]_{\mathbf{Z}}, \kappa_{[a,b]_{\mathbf{Z}}})$ (or for short $[a, b]_{\mathbf{Z}}$ if there is no danger of ambiguity) a Khalimsky interval. Furthermore, the product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, \kappa)$ is called the *Khalimsky product topology* on $\mathbf{Z}^n$ (or the *Khalimsky nD space*), denoted by $(\mathbf{Z}^n, \kappa^n)$. Hereafter, for a subset $X \subset \mathbf{Z}^n$ we will denote by (X, κ^n_X), $n \ge 1$ the subspace induced by $(\mathbf{Z}^n, \kappa^n)$, and we call it a K-topological space.

Let us now examine the structure of $(\mathbf{Z}^n, \kappa^n)$. A point $x = (x_i)_{i \in [1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ is *pure open* if all coordinates are odd, and *pure closed* if each of the coordinates is even and the other points in $\mathbf{Z}^n$ are called *mixed* [16]. These points are shown like following symbols: The symbols ■, a black jumbo dot, • mean a pure closed point, a pure open point and a mixed point (see Fig. 2), respectively.

In relation to the further statement of a mixed point in $(\mathbf{Z}^2, \kappa^2)$, for the points $p = (2m, 2n+1)$ (resp. $p = (2m+1, 2n)$), we call the point p *closed-open* (resp. *open-closed*) [26]. With this perspective, we clearly observe that for the point $p = (p_1, p_2)$ of $\mathbf{Z}^2$ the *smallest (open) neighborhood* of the point, denoted by $SN_K(p) \subset \mathbf{Z}^2$, is the following: