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# Topological dynamics on finite directed graphs

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ABSTRACT

We establish that finite directed graphs give rise to semiflows on the power set of their nodes. We analyze the topological dynamics for semiflows on finite directed graphs by characterizing Morse decompositions, recurrence behavior and attractor–repeller pairs under weaker assumptions. As is expected, the discrete metric plays an important role in our constructions and their consequences.


## 1. Introduction

The mathematical theory of dynamical systems analyzes, from an axiomatic point of view, the common features of many models that describe the behavior of systems in time. In its abstract form, a dynamical system is given by a time set $T$ (with semigroup operation $\circ$), a state space $M$, and a map $\Phi : T \times M \to M$ that satisfies (i) $\Phi(0, x) = x$ for all $x \in M$, describing the initial value, and (ii) $\Phi(t \circ s, x) = \Phi(t, \Phi(s, x))$ for all $t, s \in T$ and $x \in M$. At the heart of the theory of dynamical systems is the study of systems behavior when $t \to \pm\infty$ (qualitative behavior), as well the change in behavior under variation of parameters (bifurcation theory) [2,6–9].

In this work we consider dynamical systems on finite directed graphs without multiple edges. We analyze their communication structure, i.e., equivalence classes of vertices that can be reached mutually via sequences of edges. This leads to the set of *communicating classes* $\mathcal{C}$ of a graph and a reachability order $\preceq$ on $\mathcal{C}$. The key concept is that of an $L$-graph corresponding to graphs for which each vertex has out-degree $\geq 1$. As it turns out, these are exactly the graphs for which the $\omega$-limit sets of the associated semiflow are nonempty. To each graph $G = (V, E)$, where $V$ is the set of vertices, $\mathcal{P}(V)$ the power set of $V$, and $E \subset V \times V$ the set of edges, we associate a semiflow $\Phi_G : \mathbb{N} \times \mathcal{P}(V) \to \mathcal{P}(V)$. This semiflow is studied from the point of view of qualitative

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behavior of dynamical systems. We adapt the concepts of $\omega$-limit sets, (positive) invariance, recurrence, Morse decompositions, attractors and attractor–repeller pairs to $\Phi_G$ and prove characterizations equivalent to those in [2]. As it turns out, the finest Morse decomposition of the semiflow $\Phi_G$ corresponds to the decomposition of $G$ into the communicating classes $\mathcal{C}$. In addition, the order on the communicating classes is equivalent to the order that accompanies a Morse decomposition. Moreover, the connected components of the recurrent set of $\Phi_G$ are exactly the (finest) Morse sets of $\Phi_G$, i.e., the communicating classes of $G$.

Graphs $G = (V, E)$ are often studied using the adjacency matrix $A_G$. The products of $A_G$ describe the paths, and hence the communication structure of $G$. We construct a semiflow $\Psi_A : \mathbb{N} \times \mathcal{Q}^d \to \mathcal{Q}^d$ (where $\mathcal{Q}^d$ is the vertex set of the unit cube in $\mathbb{R}^d$) that is equivalent to the semiflow $\Phi_G$ defined on $\mathcal{P}(V)$, using Boolean matrix multiplication. This point of view is somewhat different from the standard approach that uses regular matrix multiplication and that does not lead to an equivalent semiflow. The equivalence allows us to interpret all results obtained for $\Phi_G$ in terms of certain linear iterated function systems.

## 2. Orbit decomposition of finite directed graphs

We start by presenting basic definitions and notations used along this work. Our first goal is to produce two simple graph decompositions motivated by concepts form dynamical systems. In contrast to the standard convention, throughout this work, the *vertex communication is not in necessarily an equivalence relation*. This simple observation will lead to interesting phenomena. We define communicating sets and classes, and a partial order between communicating classes is presented. We introduce a necessary condition to meaningfully study asymptotic behavior via the concept of orbit. Some of the results presented in this section are somehow elementary, we prefer to include such results for the sake of exposition.

**Remark 2.1.** Throughout this note whenever we refer to a *graph* $G = (V, E)$ we mean a *finite directed graph* when $V$ is the set of vertices in $G$ and $E$ is the set of edges in $G$ such that *no multiple edges between any two elements in $V$ are allowed*.

### *2.1. Orbits and communicating classes*

The communication structure in graphs is a central concept in this work. In this section we introduce the concepts of communicating sets and communicating classes based on the idea of orbits.

Let $G$ be a graph. An edge from the vertex $i$ to the vertex $j$ is denoted $(i, j) \in E$. A path in $G$ correspond to a sequence of vertices agree with the incidence and direction in $G$ and is denoted by $\langle i_0 i_1 \dots i_n \rangle$. Sometimes we write $i \in \gamma$ to specify that the vertex $i$ belongs to the path $\gamma$. We define the set:

$$\Gamma^n = \{\gamma : \ell(\gamma) = n, \;\; n \in \mathbb{N}\} \tag{2.1}$$

as the set of all paths $\gamma$ of *length* $\ell(\gamma) = n$, with $\Gamma^0 = V$. We can specify vertices in a path $\gamma$ in terms of the projection maps $\pi_p$ for $0 \le p \le n$:

$$\pi_p : \Gamma^n \to V, \;\; \pi_p(\gamma) = i_p$$

where $i_p$ is the $p$th vertex in $\gamma$. In other words,

$$\gamma = \langle \pi_0(\gamma) \dots \pi_p(\gamma) \dots \pi_n(\gamma) \rangle.$$

A subpath $\gamma'$ of $\gamma$ is a subsequence of $\gamma$ of consecutive edges (or vertices) belonging to $\gamma$. In particular, any edge of a path is a subpath of length one. Composition of paths will play a role in many of our proofs.