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2-dimensional polyhedra with finite depth

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ABSTRACT

In this paper every polyhedron is finite and every ANR is compact. Let $P \geq X_1 \geq X_2 \geq \ldots$, be a sequence, in which P is a polyhedron and $\geq$ are homotopy dominations. One may ask, if each sequence of this form contains only finitely many homotopy dominations that are not homotopy equivalences, or if there exists an integer l_P (independent of the sequence) that each sequence contains only $\leq l_P$ homotopy dominations that are not homotopy equivalences. Closely related open questions are: *Does there exist a polyhedron P homotopy dominating an infinite sequence of polyhedra $\{P_i\}$, where P_i homotopy dominates P_{i+1} but P_i and P_{i+1} have different homotopy types, for every $i \in N$?* (M. Moron) [30, Problem 1436], and the famous problem of K. Borsuk (1967): *Is it true that two ANR's homotopy dominating each other have the same homotopy type?* Here we prove that, if $\dim P = 2$, then the answers to all these questions depend only on the properties of the fundamental group of P (for 1-dimensional polyhedra, the answers are obvious). Furthermore, if sequences in consideration contain only polyhedra, then, for the positive answer it suffices to answer positively the analog of our topological question for finitely presented groups (the fundamental groups) with retractions. Applying these results, we prove that for each polyhedron P with $\dim P \leq 2$ and elementary amenable fundamental group G with cd$G < \infty$, there is a bound l_P on the lengths of all descending sequences $P \geq X_1 \geq X_2 \ldots$ of homotopy dominations that are not homotopy equivalences. The same holds if the fundamental group of P is a limit group. It means that such polyhedra P have finite depth.

1. Introduction

In this paper every polyhedron is finite and every ANR is compact. We assume (without loss of generality) that every polyhedron, ANR, $FANR$ and CW-complex is connected.

Recall that a *domination* in a given category $\mathcal{C}$ is a morphism $f : X \to Y$, $X, Y \in Ob\mathcal{C}$, for which there exists a morphism $g : Y \to X$ of $\mathcal{C}$ such that $fg = id_Y$. Then we say that Y is dominated by X, and we write $Y \leq X$, or $X \geq Y$.

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We write that $X < Y$ iff $X \leq Y$ holds but $Y \leq X$ fails.

In 1968 K. Borsuk asked (in the shape-theoretical language): *Is it true that every polyhedron homotopy dominates only finitely many different homotopy types?* [4].

We showed in [24] that the answer to this question is negative — there exists a finite polyhedron (even 2-dimensional) dominating infinitely many different homotopy types (of 2-dimensional polyhedra). Furthermore, counterexamples to Borsuk's question are possible even with polycyclic fundamental groups [25,22].

Now, one can consider sequences of spaces homotopy dominated by a given polyhedron P with a quasi-order determined by the relation of homotopy domination $P \geq X_1 \geq X_2 \geq \ldots$. Recall that, by the classical results of J.H.C. Whitehead and C.T.C. Wall, each topological space homotopy dominated by a polyhedron has the homotopy type of a CW-complex, but not necessarily finite [31].

Let us state the following question:

Problem 1. Does there exist a sequence $P \geq X_1 \geq X_2 \geq \ldots$, where P is a polyhedron, containing infinitely many homotopy dominations that are not homotopy equivalences?

This problem is closely related to other natural open questions published in the literature. One of them is the famous problem of K. Borsuk posed in 1967, in his monograph "Theory of Retracts" [6, Ch. IX, Problem (12.7)]:

Problem 2. Is it true that two $ANR's$ homotopy dominating each other have the same homotopy type?

Since, by the result of J. West [33], every ANR has the homotopy type of a polyhedron, this question is equivalent to its analog for polyhedra. In [13] the same problem was stated for $FANR's$, compacta which are generalizations of ANR's, and correspond in the shape category to CW-complexes dominated by polyhedra in the homotopy category (see below). Recall that on ANR's shape and homotopy theory coincide (see [29], [14], [5]).

An other problem of the similar nature was posed by M. Moron and published in [30, Problem 1436 (or Problem 9), p. 672]:

Problem 3. Is there a polyhedron P dominating a sequence of polyhedra $\{P_i\}$ where P_i homotopy dominates P_{i+1}, but P_i and P_{i+1} have different homotopy types, for every $i \in N$?

In the 1970's K. Borsuk introduced the following notion of *depth* (in the shape category of compacta, see [4]): A system $X_1 < X_2 < \ldots < X_k \leq A$, where $X_i \in Ob\mathcal{C}$, for $i = 1, 2, \ldots, k$, is called a *chain of length* k for $A \in Ob\mathcal{C}$. The *depth* $D(A)$ of A is the least upper bound of the lengths of all chains for A. If this upper bound is infinite, we write $D(A) = \aleph_0$.

We will consider depth in the homotopy category of CW-complexes. However, there is a 1–1 functorial correspondence between the shapes of compacta shape dominated by a given polyhedron and the homotopy types of CW-complexes homotopy dominated by it (see [14, Theorem 2.2.6]; [15]).

Moreover, the depth of a given polyhedron is the same in both, pointed and unpointed, cases (it follows from [12, Theorem 5.1]).

Clearly, the following problem is also closely related to Problems 1, 2 and 3:

Problem 4. Does there exist a polyhedron P with infinite depth $D(P)$?

In this paper we study 2-dimensional polyhedra. In dimension 1 the answers to all the above questions are obvious. This follows from the fact that every 1-dimensional polyhedron has the homotopy type of a finite wedge of the circles S^1, hence (as a $K(G, 1)$) space homotopy dominates just over the polyhedra of the same kind (with not greater Betti numbers).