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A study on convergence and ideal convergence classes

D.N. Georgiou [a], A.C. Megaritis [b,*], G.A. Prinos [a]

[a] University of Patras, Department of Mathematics, 265 00 Patras, Greece
[b] Technological Educational Institute of Western Greece, Department of Accounting and Finance, 302 00 Messolonghi, Greece

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ABSTRACT

Let X be a non-empty set. We introduce semi-convergence classes on X in order to obtain a modification of classical Kelley's theorem. Subsequently, we do some further investigations on ideal convergence classes (see [7]). Finally, we introduce ideal semi-convergence classes $\mathcal{C}'$ on X, in order to ensure the existence of a unique topology τ on X such that: a net $(s_d)_{d\in D}$ $\mathcal{I}$-semi-convergences ($\mathcal{C}'$) to $x \in X$ i.e. $((s_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}'$, where $\mathcal{I}$ is an ideal of D, if and only if $(s_d)_{d\in D}$ $\mathcal{I}$-converges to x relative to the topology τ.

1. Introduction

Nets and filters are two different theories of convergence, which both successfully generalize the convergence of sequences. The development of a convergence theory via nets was first introduced by E. H. Moore [16,17] and in [18], jointly with H. L. Smith. Later G. Birkhoff [1] and subsequently J. W. Tukey [24] gave foundational and systematic treatment of the concept. Finally, J. L. Kelley [8,9] made refinements in the theory of Moore–Smith convergence, extended and simplified previous results. He gave a characterization of convergence of nets in arbitrary topological spaces in the following manner: Given a class $\mathcal{C}$ of pairs (s, x), where s is a net in X and x is a point of X, he found conditions on this class that guarantee the existence of a unique topology τ on X, such that $(s, x) \in \mathcal{C}$ if and only if s converges to x, relative to the topology τ.

On the other side the concept of convergence of a sequence has been extended to statistical convergence, with the assistance of the natural density of subsets of the set $\mathbb{N}$ of positive integers. Statistical convergence

* Corresponding author.
E-mail addresses: georgiou@math.upatras.gr (D.N. Georgiou), thanasismeg13@gmail.com (A.C. Megaritis), gprinos161168@yahoo.gr (G.A. Prinos).

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was introduced by H. Fast [5], I. J. Schoenberg [22], H. Steinhaus [23], and A. Zygmund [25] and soon became an active area of research. J. A. Fridy [6], T. Šalát [20], G. Di Maio and L. D. R. Kočinac [14] and many others made substantial contributions to the theory. Applications of statistical convergence in mathematical analysis and the theory of numbers can be found in [2–4,14,15].

The concept of $\mathcal{I}$-convergence of a sequence of real numbers, where $\mathcal{I}$ is an ideal on the natural numbers, was first introduced in [10] and it is based on the notion of the ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$. Later it was extended to general spaces in [11,12,19,21]. The $\mathcal{I}$-convergence is a natural generalization of usual convergence and statistical convergence of sequences. B. K. Lahiri and P. Das in [12,13] defined and examined the notion of $\mathcal{I}$-convergence of sequences and nets in general topological spaces.

In this paper, we introduce and study the concepts of semi-convergence classes and ideal semi-convergence classes. Specifically, in Section 2 we give some preliminaries which will be used later. In Section 3 we define the notion of a semi-convergence class and we obtain a modification of classical Kelley's theorem. In Section 4 we present propositions concerning the ideal convergence classes. Finally, in Section 5 we define the notion of an ideal semi-convergence class in order to modify the ideal convergence classes theorem (see [7]).

2. Preliminaries

In this section, we recall some of the basic concepts related to the usual convergence and ideal convergence of nets in topological spaces and we refer to [9] and [13] for more details.

Let D be a non-empty set. A non-empty family $\mathcal{I}$ of subsets of D is called *ideal* if $\mathcal{I}$ has the following properties:

(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(2) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called *proper* if $D \notin \mathcal{I}$.

A partially preordered set D is called *directed* if every two elements of D have an upper bound in D.

If $(D, \leqslant_D)$ and $(E, \leqslant_E)$ are directed sets, then the Cartesian product $D \times E$ is directed by $\leqslant$, where $(d_1, e_1) \leqslant (d_2, e_2)$ if and only if $d_1 \leqslant_D d_2$ and $e_1 \leqslant_E e_2$. Also, if $(E_d, \leqslant_d)$ is a directed set for each d in a set D, then the product

$$\prod_{d\in D} E_d = \{f : D \to \bigcup_{d\in D} E_d : f(d) \in E_d \text{ for all } d \in D\}$$

is directed by $\leqslant$, where $f \leqslant g$ if and only if $f(d) \leqslant_d g(d)$, for all $d \in D$.

A net in a set X is an arbitrary function s from a non-empty directed set D to X. If $s(d) = s_d$, for all $d \in D$, then the net s will be denoted by the symbol $(s_d)_{d\in D}$.

A net $(t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *semisubnet* of the net $(s_d)_{d\in D}$ in X if there exists a function $\varphi : \Lambda \to D$ such that $t = s \circ \varphi$. We write $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ to indicate the fact that φ is the function mentioned above.

A net $(t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *subnet* of the net $(s_d)_{d\in D}$ in X if there exists a function $\varphi : \Lambda \to D$ with the following properties:

(1) $t = s \circ \varphi$, or equivalently, $t_\lambda = s_{\varphi(\lambda)}$ for every $\lambda \in \Lambda$.
(2) For every $d \in D$ there exists $\lambda_0 \in \Lambda$ such that $\varphi(\lambda) \geqslant d$, whenever $\lambda \geqslant \lambda_0$.

Suppose that $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ is a semisubnet of the net $(s_d)_{d\in D}$ in X. For every ideal $\mathcal{I}$ of the directed set D, we consider the family $\{A \subseteq \Lambda : \varphi(A) \in \mathcal{I}\}$. This family is an ideal of Λ which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

We say that a net $(s_d)_{d\in D}$ *converges* to a point $x \in X$ if for every open neighborhood U of x there exists a $d_0 \in D$ such that $s_d \in U$ for all $d \geqslant d_0$. In this case we write $\lim_{d\in D} s_d = x$.