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Products of topological groups in which all closed subgroups are separable

Arkady G. Leiderman [a,1,2], Mikhail G. Tkachenko [b,*,2]

[a] *Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva, P.O.B. 653, Israel*
[b] *Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Del. Iztapalapa, C.P. 09340, Mexico City, Mexico*

A R T I C L E I N F O

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To the memory of Wistar Comfort (1933–2016), a great topologist and man, to whom we owe much of our inspiration

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We prove that if H is a topological group such that all closed subgroups of H are separable, then the product $G \times H$ has the same property for every separable compact group G.

Let $\mathfrak{c}$ be the cardinality of the continuum. Assuming $2^{\omega_1} = \mathfrak{c}$, we show that there exist:

- pseudocompact topological abelian groups G and H such that all closed subgroups of G and H are separable, but the product $G \times H$ contains a closed non-separable σ-compact subgroup;
- pseudocomplete locally convex vector spaces K and L such that all closed vector subspaces of K and L are separable, but the product $K \times L$ contains a closed non-separable σ-compact vector subspace.

1. Introduction

All topological groups and locally convex linear spaces are assumed to be Hausdorff. The *weight* of a topological space X, denoted by $w(X)$, is the smallest size of a base for X. A space X is *separable* if it contains a dense countable subset. If every subspace of a topological space X is separable, then X is called

* Corresponding author.
E-mail addresses: arkady@math.bgu.ac.il (A.G. Leiderman), mich@xanum.uam.mx (M.G. Tkachenko).

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hereditarily separable. Hereditary separability is not a productive property — the Sorgenfrey line is an example of a hereditarily separable paratopological group whose square contains a closed discrete subgroup of cardinality $\mathfrak{c}$ (see [6, 2.3.12] or [1, 5.2.e]). Nevertheless, as we observe in Proposition 1.2, the product of any hereditarily separable topological space with a separable metrizable space is hereditarily separable.

Our main objective is to study products of two topological groups having the following property: Every *closed subgroup* of a group is separable. Since this property does not imply the separability of every subspace of a group, Proposition 1.2 has very limited applicability for our purposes.

It is known that a closed subgroup of a separable topological group is not necessarily separable. However, W. Comfort and G. Itzkowitz proved in [3] that all closed subgroups of a separable locally compact topological group are separable. It was also noticed by several authors independently that every metrizable subgroup of a separable topological group is separable (see [12]).

Recently these results have been generalized in [11] as follows: Every feathered subgroup of a separable topological group is separable. We recall that a topological group G is *feathered* if it contains a compact subgroup K such that the quotient space G/K is metrizable (see [1, Section 4.3]). All locally compact and all metrizable groups are feathered.

Since the class of feathered groups is closed under countable products and taking closed subgroups, we obtain the following simple corollary.

Proposition 1.1. *Let G be a separable locally compact group and H be a separable feathered group. Then every closed subgroup of the product $G \times H$ is separable.*

Let us say that a topological group G is *strongly separable* (briefly, *S-separable*), if for any topological group H such that every closed subgroup of H is separable, the product $G \times H$ has the same property.

The following open problem arises naturally.

Problem 1. Find out the frontiers of the class of S-separable topological groups:

(a) Is every separable locally compact group S-separable?
(b) Is the group of reals $\mathbb{R}$ S-separable? Does there exist a separable metrizable group which is not S-separable?
(c) Is the free topological group on the closed unit interval S-separable?

Our Theorem 2.1 provides the positive answer to (a) of Problem 1 in the important case when G is a separable compact group.

Then we deduce that every topological group G which contains a separable compact subgroup K such that the quotient space G/K is countable, is S-separable.

It is reasonable to ask whether the separability of closed subgroups of the product $G \times H$ is determined by the same property of the factors G and H, without imposing additional conditions on G or H. We answer this question in the negative in Section 3.

A Tychonoff space X is called *pseudocompact* if every continuous real-valued function defined on X is bounded. Assuming that $2^{\omega_1} = \mathfrak{c}$, we construct in Theorem 3.4 pseudocompact topological abelian groups G and H such that all closed subgroups of G and H are separable, but the product $G \times H$ contains a closed non-separable σ-compact subgroup.

In Section 4 we consider the class of locally convex spaces (lcs) in which all closed vector subspaces are separable. The case of locally convex spaces is quite different from topological groups, as an infinite-dimensional lcs is never locally compact or pseudocompact. Probably the first example of a closed (but not complete) non-separable vector subspace of a separable lcs was given by R. Lohman and W. Stiles [12]. The study of the products of topological vector spaces in which all closed vector subspaces are separable