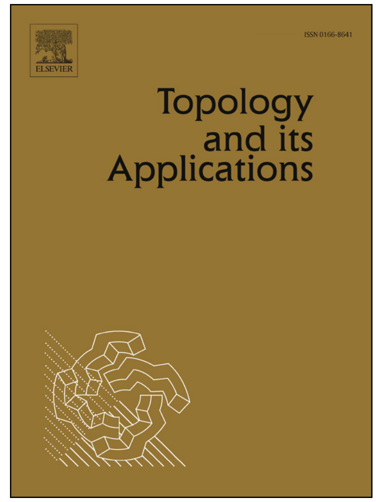

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M. Isabel Garrido, Ana S. Meroño

PII: S0166-8641(18)30212-8
DOI: https://doi.org/10.1016/j.topol.2018.03.033
Reference: TOPOL 6452

To appear in: *Topology and its Applications*

Received date: 31 December 2016
Revised date: 4 December 2017
Accepted date: 5 February 2018

Please cite this article in press as: M. Isabel Garrido, A.S. Meroño, The Samuel realcompactification, *Topol. Appl.* (2018), https://doi.org/10.1016/j.topol.2018.03.033

THE SAMUEL REALCOMPACTIFICATION

M. ISABEL GARRIDO AND ANA S. MEROÑO*

Abstract. For a uniform space (X, μ), we introduce a realcompactification of X by means of the family $U_\mu(X)$ of all the real-valued uniformly continuous functions on X, in the same way that the known Samuel compactification of the space is given by $U_\mu^*(X)$ the set of all the bounded functions in $U_\mu(X)$. We will call it "the Samuel realcompactification" by several resemblances to the Samuel compactification. In this paper, we present different ways to construct such realcompactification as well as we study the corresponding problem of knowing when a uniform space is Samuel realcompact, that is, when it (topologically) coincides with its Samuel realcompactification. At this respect, we obtain as main result a theorem of Katětov-Shirota type, given in terms of a property of completeness, recently introduced by the authors, called Bourbaki-completeness.

1. Introduction

A realcompactification of a Tychonoff space X is a realcompact space Y in which X is densely embedded. For instance, the well-known Hewitt-Nachbin realcompactification υX. Recall that υX is characterized as the smallest realcompactification of X (in the usual order on the family of all realcompactifications) such that every real-valued continuous function $f \in C(X)$ can be continuously extended to it [9]. In the frame of uniform spaces, since we can also consider $U_\mu(X)$, the set of all the real-valued uniformly continuous functions on the uniform space (X, μ), it is natural to ask what is the smallest realcompactification of X such that every function $f \in U_\mu(X)$ can be continuously extended to it. Here, following the ideas of [3], we will introduce this realcompactification that we denote by $H(U_\mu(X))$ because in fact it can be represented as the set of all the real homomorphisms on the unital vector lattice $U_\mu(X)$. Moreover, we will see that it also coincides, as a topological space, with the completion of the uniform space $(X, w_{U_\mu(X)})$, where $w_{U_\mu(X)}$ denotes the weak uniformity on X generated by $U_\mu(X)$ (Theorem 1).

We will call $H(U_\mu(X))$ the *Samuel realcompactification* of (X, μ) in likeness to the Samuel compactification $s_\mu X$. Recall that the Samuel compactification of a uniform space (X, μ), also known as the Smirnov compactification, is the smallest (real)compactification of X such that every bounded real-valued uniformly continuous function $f \in U_\mu^*(X)$ can be continuously extended to it ([20]). The resemblance between the Samuel realcompactification and the Samuel compactification is not only due to their characterization as the smallest realcompactification, respectively compactification, such that every real-valued, respectively bounded, uniformly continuous function can be continuously extended to it. In fact, as well as every compactification of a Tychonoff space X can be considered as a Samuel compactification for some totally bounded (precompact) uniformity on X [2] (see also [10]), every realcompactification of X can be considered as a Samuel realcompactification for some uniformity on it, as we will explain below (Theorem 2).

A Tychonoff space X is realcompact whenever $X = \upsilon X$, and we can similarly define, for a uniform space (X, μ), that (X, μ) is *Samuel realcompact* if $X = H(U_\mu(X))$. In this paper, we give a uniform analogue to the well-known Katětov-Shirota Theorem ([21], see also [9]) which deals with realcompactness of Tychonoff spaces. This classical theorem states that *a Tychonoff space X is realcompact if and only if X is completely uniformizable and every closed discrete subspace has non-measurable cardinal*. In this line, we will obtain an analogous result that asserts that *a uniform space is Samuel realcompact if and only if it is Bourbaki-complete and every (closed) uniformly discrete subspace has non-measurable cardinal* (Theorem 12). The property of Bourbaki-completeness was introduced and studied by the authors in [5] (see also [6] and [12]), and it is a uniform property stronger than usual completeness. We say that a uniform space is *Bourbaki-complete* if every *Bourbaki-Cauchy* filter, defined later, clusters.

2010 *Mathematics Subject Classification.* Primary 54D60, 54E15; Secondary 54D35, 54A20, 54A25.

Key words and phrases. Uniform space, realcompactification, real-valued uniformly continuous function, Samuel realcompactification, Cauchy filter, Bourbaki-Cauchy filter, Bourbaki-completeness.

*Corresponding author.

Partially supported by MINECO Project-MTM2015-65825-P (Spain).