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Splitting ultra-metrics by T_0-ultra-quasi-metrics ☆

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A B S T R A C T

Given a T_0-ultra-quasi-metric u on a set X, we write u^s for its symmetrization $u \vee u^{-1}$. In this paper we show that there exists a T_0-ultra-quasi-metric v on X such that $v \leq u$, $v^s = u^s$ and the specialization order of v is linear.
We also discuss connections of this statement with related results about Robinsonian dissimilarities and orderability of ultra-metrizable topological spaces.

1. Introduction

The studies on ultra-quasi-pseudometrics in the present paper are related to the investigations completed in [2,3] by Gaba and Künzi on arbitrary quasi-pseudometrics. In [2,3] a partially ordered metric space $(X, m, \leq)$ was said to be *produced* by a T_0-quasi-metric d on X provided that the specialization order of d is equal to $\leq$ and $d^s = d \vee d^{-1} = m$. In this way the authors obtained a (metric) theory of partially ordered metric spaces that is closely related to Nachbin's theory of uniform ordered topological spaces [7].

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In this paper we shall study similarly (partially) ordered ultra-metric spaces $(X, m, \leq)$ such that there is a T_0-ultra-quasi-metric d on X the specialization order of which is equal to $\leq$ and $d^s = m$.

While (see Section 3) there are obvious similarities between the methods discussed in [2,3] and the techniques developed in this paper, in Section 4 some striking differences between the two theories will become apparent.

Somewhat in the spirit of Szpilrajn's Theorem (compare [2, Section 3]) the main result of this paper shows that given a T_0-ultra-quasi-metric u on a set X, there exists a T_0-ultra-quasi-metric v on X such that $v \leq u$, $v^s = u^s$ and the specialization order of v is linear.

As a contrast, let us recall two facts mentioned by Gaba and Künzi:

(1) In [3, Example 6] an example of a *metric* space (X, m) is given for which there does not exist a T_0-quasi-metric d on X such that the induced topologies $\tau(d^s)$ and $\tau(m)$ are equal and the specialization order of d linear.
(2) In [2, Example 5] the authors give an example of a finite *metric* space (X, m) such that there exists a T_0-*quasi-metric* d on X that is minimal among those T_0-quasi-metrics t on X satisfying $t^s = m$, but does not have a linear specialization order (compare also Example 6 below).

Throughout we discuss connections of our result with some other statements from the mathematical literature, belonging mainly to the theory of Robinsonian dissimilarities [11] and the theory of orderability of ultra-metrizable topological spaces [4].

2. Preliminaries

We first recall basic definitions from asymmetric topology (see for instance [1,5]; for quasi-pseudometrics and metrics see also [2,3]).

Definition 1. Let X be a set and $d : X \times X \to [0, \infty)$ a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then d is an *ultra-quasi-pseudometric* on X if

(a) $d(x, x) = 0$ whenever $x \in X$, and
(b) $d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(x, y) \vee d(y, z)$ whenever $x, y, z \in X$.

We shall say that (X, d) is a T_0-*ultra-quasi-metric space* provided that d also satisfies the following condition: For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$.

Let d be an ultra-quasi-pseudometric on a set X. Then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also an ultra-quasi-pseudometric on X, called the *conjugate* or *dual ultra-quasi-pseudometric* of d. Observe that if d is a T_0-ultra-quasi-metric on X, then $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is an ultra-metric on X.

Recall that if d is an ultra-quasi-pseudometric on X, then the binary relation $\leq$ on X defined by $x \leq_d y$ if and only if $d(x, y) = 0$ defines a partial preorder on X, which is called the *specialization (partial) preorder* of d on X. Note that $\leq_d$ is a partial order if and only if d satisfies the T_0-condition.

We partially order the set $U(X)$ of all ultra-quasi-pseudometrics on a set X by setting $s \leq t$ if $s(x, y) \leq t(x, y)$ whenever $x, y \in X$.

Definition 2. Given a metric space (X, m) we shall call a quasi-pseudometric d on X *m-splitting* provided that $d^s = d \vee d^{-1} = m$. An m-splitting quasi-pseudometric d on X is called *minimally m-splitting* provided that whenever e is a quasi-pseudometric on X with $e \leq d$ and $e^s = m$, then $e = d$.