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Remarks on selectively absolute star-Lindelöf spaces ☆

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ABSTRACT

A space X is *selectively absolutely star-Lindelöf* [1,3] if for each open cover $\mathcal{U}$ of X and any sequence $(D_n : n \in \omega)$ of dense subsets of X, there are finite sets $F_n \subseteq D_n (n \in \omega)$ such that $St(\bigcup_{n\in\omega} F_n, \mathcal{U}) = X$. In this paper, we continue to investigate topological properties of selectively absolute star-Lindelöf spaces, and show the following statements:

(1) There exists a Tychonoff selectively a-star-Lindelöf, pseudocompact space X having a regular closed G_δ subset which is not star-Lindelöf (hence not selectively a-star-Lindelöf);
(2) Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there exists a normal selectively a-star-Lindelöf space X having a regular closed G_δ subset which is not star-Lindelöf (hence not selectively a-star-Lindelöf);
(3) An open F_σ-subset of a selectively a-star-Lindelöf space is selectively a-star-Lindelöf;
(4) For any cardinal κ, there exists a Tychonoff selectively a-star-Lindelöf, pseudocompact space X such that $e(X) \geq \kappa$.

1. Introduction

By a space, we mean a topological space. In this section, we give definitions of terms which are used in this paper. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $St(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $St(x, \mathcal{U})$ instead of $St(\{x\}, \mathcal{U})$.

Recall that a space X is *star-Lindelöf* if for each open cover $\mathcal{U}$ of X, there is a countable set $C \subseteq X$ such that $St(C, \mathcal{U}) = X$ (see for example [4,8], where different terminology is used). Clearly every separable

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space is star-Lindelöf. It is well-known that every T_2 space X is countably compact if and only if for each open cover $\mathcal{U}$ of X, there is a finite set $F \subseteq X$ such that $St(F, \mathcal{U}) = X$ ([4]). Thus every T_2 countably compact space is star-Lindelöf.

Definition 1.1. [2] A space X is *absolutely star-Lindelöf* (briefly, a-star-Lindelöf) if for any open cover $\mathcal{U}$ of X and any dense subset D of X, there exists a countable subset $C \subseteq D$ such that $X = St(C, \mathcal{U})$.

Obviously, every a-star-Lindelöf space is star-Lindelöf.

Definition 1.2. [6] A space X is *absolutely countably compact* (briefly, acc) if for any open cover $\mathcal{U}$ of X and any dense subset D of X, there exists a finite subset $F \subseteq D$ such that $X = St(F, \mathcal{U})$.

Clearly, a compact space is acc, and an acc T_2 space is countably compact. It is known that a space is acc if and only if it is countably compact and a-star-Lindelöf (see [2, Proposition 1.6]).

Recently, Bhowmik [1] introduced the following notion as a variation of star-Lindelöfness.

Definition 1.3. [1] A space X is *selectively absolutely star-Lindelöf* (briefly, selectively a-star-Lindelöf) if for each open cover $\mathcal{U}$ of X and any sequence $(D_n : n \in \omega)$ of dense subsets of X, there are finite sets $F_n \subseteq D_n (n \in \omega)$ such that $St(\bigcup_{n\in\omega} F_n, \mathcal{U}) = X$.

By the definitions above, it is clear that every selectively a-star-Lindelöf space is a-star-Lindelöf and every a-star-Lindelöf space is star-Lindelöf.

In [3], Bonanzinga, Cuzzupé and Sakai studied the relationship between selectively a-star-Lindelöf spaces and related spaces, and topological properties of selectively a-star-Lindelöf spaces. The purpose of this note is to continue to investigate topological properties of selectively a-star-Lindelöf spaces.

Throughout the paper, the cardinality of a set A is denoted by $|A|$. Let $\mathfrak{c}$ denote the cardinality of the continuum, ω_1 the first uncountable cardinal and ω the first infinite cardinal. For a cardinal κ, let κ^+be the smallest cardinal greater than κ. For a pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define can be found in [5].

2. Properties of selectively a-star-Lindelöf spaces

In [3], Bonanzinga, Cuzzupé and Sakai showed that a regular closed subset of a selectively a-star-Lindelöf space need not be selectively a-star-Lindelöf. In the following, we give a stronger example showing that a regular closed G_δ subset of a Tychonoff selectively a-star-Lindelöf, pseudocompact space need not be selectively a-star-Lindelöf. For the next example, we need the following Lemma.

Lemma 2.1. *Let X be a space with a dense set of isolated points. Then X is selectively a-star-Lindelöf if and only if it is a-star-Lindelöf.*

Proof. Let D be a dense set of isolated points of X. Then every dense subset of X contains D. We only show that if X is a-star-Lindelöf then X is selectively a-star-Lindelöf. Let $\mathcal{U}$ be an open cover of X and $(D_n : n \in \omega)$ be a sequence of dense subsets of X. Then there exists a countable subset $C \subseteq D$ such that $St(C, \mathcal{U}) = X$, since X is a-star-Lindelöf. Let $C = \{d_n : n \in \omega\}$. For each $n \in \omega$, let $F_n = \{d_n\}$. Then F_n is finite and $F_n \subseteq D_n$ for each $n \in \omega$, since each D_n contains D. Hence $St(\bigcup_{n\in\omega} F_n, \mathcal{U}) = St(C, \mathcal{U}) = X$. Thus we complete the proof. □