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On non-cut points and COTS

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ABSTRACT

The concept of cut point convex sets is used to study non-cut points of a connected topological space. Some relations between cut point convex sets, H-sets and COTS are established. We prove that an H-set of a COTS is contained in a COTS with endpoints a and b for some a, b in the H-set. It is shown that if a connected topological space X has at most two non-cut points and an $R(i)$ set that contains all the closed cut points of X, then X is a COTS with endpoints. Further we show that if every non-degenerate proper regular closed connected subset of a connected topological space X contains only finitely many closed points of X, then X has at least two non-cut points. A characterization of a non-indiscrete finite connected subspace of the Khalimsky line is also obtained.

1. Introduction

The concept of connected ordered topological space (COTS), introduced by Khalimsky, Kopperman and Meyer in [8] does not require any separation axiom. In view of its applications in theoretical computer science, the study of COTS is gaining momentum ([8]). The concepts of cut points and non-cut points form an integral part of any consideration of COTS. Topological spaces are assumed to be connected for any study regarding COTS. By Proposition 2.5 of [8], a COTS has at most two non-cut points, which turn out to be endpoint(s) in each of its two orders. Thus there comes the concept of COTS with endpoints. One of the directions of study of COTS is to know about subcategories of the category TOP (consisting of all topological spaces) whose objects become COTS with or without endpoints. Papers [3], [4], [5] and [6]

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introduce some subclasses of TOP, whose members are COTS with endpoints. The set of the integers $\mathbb{Z}$, equipped with the topology whose base is $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\} \cup \{\{2n+1\} : n \in \mathbb{Z}\}$ is the Khalimsky line ([1]) which is also a COTS but not T_1. Each non-indiscrete finite COTS is homeomorphic to a finite connected subspace of the Khalimsky line ([8]). A cut point convex set of a topological space defined in [6] becomes connected if the space is a COTS ([6, Proposition 3.1(iii)]).

In this paper, we study non-cut points and COTS in some subclasses of TOP and involve H-sets, $R(i)$ sets, wherever necessary. Notation, definitions and preliminaries are given in Section 2. The main results of the paper appear in Sections 3, 4 and 5. In Section 3, we note that in a connected space X with endpoints, there is no proper cut point convex set of X containing all non-cut points. We prove that if X is a connected space, $H \subset X$ is a non-degenerate H-set and $Y \subset X$ a non-empty cut point convex set such that $\phi \neq (H - Y) \subset \text{ct}X$, then there is some $q \in H - Y$ such that $H \subset A_q(Y) \cup \{q\}$ where $A_q(Y)$ is the separating set of $X - \{q\}$ which contains Y, and deduce that for each $z \in H$ with $H - \{z\} \subset \text{ct}X$, there exist distinct points p and q of H such that $H \subset S[p, q] \subset A_q(p) \cup \{q\}$, or $H \subset (A_q(z) \cup \{q\}) \cap (A_p(S[z, q]) \cup \{p\})$, where A_p and A_q are the chosen separating set of $X - \{p\}$ and $X - \{q\}$ respectively. Using this result, we prove that if $H \subset X$ is a non-degenerate H-set, consisting of cut points of a connected space X, then for each $z \in H$, there exist distinct points p and q of H such that $H \subset S[p, q] \subset (A_q(p) \cup \{q\}) \cap (A_p(q) \cup \{p\})$, or $H \subset (A_q(z) \cup \{q\}) \cap (A_p(S[z, q]) \cup \{p\})$, where A_p and A_q are the chosen separating set of $X - \{p\}$ and $X - \{q\}$, respectively. This considerably strengthens Theorem 27 of [9] (which is proved by assuming the space to be a cut point space, and, therefore, the H-set under consideration, has to be a proper set, because the whole set of a cut point space is not an H-set ([2]). Further we prove that every H-set of a COTS has first and last element in each of the two orders of the COTS, a similar result is proved for a compact set in [8, Lemma 2.11(ii)]. In Section 4, we show that if a connected space X has an $R(i)$ subset H such that there is no proper regular closed, connected subset of X containing H, then there is no proper cut point convex set of X containing all the non-cut points of X. This strengthens Theorem 3.9 of [6]. If a connected space X has at most two non-cut points and an $R(i)$ subset which contains $\text{cd}(X) \cap \text{ct}X$ where $\text{cd}(X)$ denotes the set of all closed points of X, then X is a COTS with endpoints. Further it is shown that if such a space is T_1 locally connected and separable, then it is homeomorphic to the closed unit interval. In Section 5, we prove that if every non-degenerate proper regular closed connected subset of a connected topological space X contains only finitely many closed points of X, then X has at least two non-cut points. This proves that if such a non-indiscrete space X has at most two non-cut points, then X is homeomorphic to a finite connected subspace of the Khalimsky line.

2. Notation, definitions and preliminaries

For notation and definitions, we shall mainly follow [6]. For completeness, we have included some of the standard notation and definitions. By space we mean topological space. A space is assumed to have at least three elements.

Let X be a space. For $H \subset Y \subset X$, $\mathbf{cl}_{\boldsymbol{Y}}(\boldsymbol{H})$ and $\mathbf{int}_{\boldsymbol{Y}}(\boldsymbol{H})$ respectively denote the closure and interior of H in Y. X is called $\boldsymbol{H(i)}$ ([11]) if every open cover of X has a finite subcollection such that the closures of the members of that subcollection cover X. A filter base γ on X is said to be **fixed** if $\cap\{A : A \in \gamma\} \neq \phi$. An **o-filter base** on X is a filter base whose members are open subsets of X. An o-filter base γ is a **regular o-filter base** if each member of γ contains the closure of some member of γ. X is called $\boldsymbol{R(i)}$ ([11]) if every regular o-filter base on X is fixed. Let $Y \subset X$. Y is $\boldsymbol{H(i)}$ ($\boldsymbol{R(i)}$) if Y, as a subspace of X, is $H(i)$ ($R(i)$). A collection α of subsets of X is a **cover** of Y in X, if $Y \subset \cup\{A : A \in \alpha\}$. Y is an $\boldsymbol{H}$**-set** ([10], [12]), if whenever a collection α consisting of open sets of X is a cover of Y in X, then there exists a finite subfamily α' of α such that $Y \subset \cup\{\text{cl}_X(G) : G \in \alpha'\}$.

Let X be a space. A point $x \in X$ is a **closed point (open point)** of X ([1]) if $\{x\}$ is closed (open) in X. The set of all closed points of X is denoted by $\text{cd}(X)$. Let $x \in X$, x is called a **cut point** if there exists