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Variations on known and recent cardinality bounds

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ABSTRACT

Sapirovskii [16] proved that $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$, for a regular space X. We introduce the *θ-pseudocharacter* of a Urysohn space X, denoted by $\psi_\theta(X)$, and prove that the previous inequality holds for Urysohn spaces replacing the bounds on cellularity $c(X) \leq \kappa$ and on pseudocharacter $\psi(X) \leq \kappa$ with a bound on Urysohn cellularity $Uc(X) \leq \kappa$ (which is a weaker condition because $Uc(X) \leq c(X)$) and on θ-pseudocharacter $\psi_\theta(X) \leq \kappa$ respectively (note that in general $\psi(\cdot) \leq \psi_\theta(\cdot)$ and in the class of regular spaces $\psi(\cdot) = \psi_\theta(\cdot)$). Further, in [6] the authors generalized the Dissanayake and Willard's inequality: $|X| \leq 2^{aL_c(X)\chi(X)}$, for Hausdorff spaces X [21], in the class of n-Hausdorff spaces and de Groot's result: $|X| \leq 2^{hL(X)}$, for Hausdorff spaces [10], in the class of T_1 spaces (see Theorems 2.22 and 2.23 in [6]). In this paper we restate Theorem 2.22 in [6] in the class of n-Urysohn spaces and give a variation of Theorem 2.23 in [6] using new cardinal functions, denoted by $UW(X)$, $\psi w_\theta(X)$, θ-$aL(X)$, $h\theta$-$aL(X)$, θ-$aL_c(X)$ and θ-$aL_\theta(X)$. In [5] the authors introduced the *Hausdorff point separating weight of a space* X denoted by $Hpsw(X)$ and proved a Hausdorff version of Charlesworth's inequality $|X| \leq psw(X)^{L(X)\psi(X)}$ [7]. In this paper, we introduce the *Urysohn point separating weight of a space* X, denoted by $Upsw(X)$, and prove that $|X| \leq Upsw(X)^{\theta\text{-}aL_c(X)\psi(X)}$, for a Urysohn space X.

1. Introduction

We shall follow notations from [11] and [13]. Recall that a space X is *Urysohn* if for every two distinct points x, $y \in X$ there are open sets U and V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.

For a space X, we denote by $\chi(X)$ (resp., $\psi(X)$, $\pi\chi(X)$, $c(X)$, $t(X)$) the *character*, (resp., *pseudocharacter*, *π-character*, *cellularity*, *tightness*) of a space X [11].

The *θ-closure* of a set A in a space X is the set $cl_\theta(A) = \{x \in X$: for every neighborhood $U \ni x$, $\overline{U} \cap A \neq \emptyset\}$; A is said to be θ-closed if $A = cl_\theta(A)$ [20]. Considering the fact that the θ-closure operator is not in general idempotent, Bella and Cammaroto defined in [2] the *θ-closed hull* of a subset A of a space X,

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denoted by $[A]_\theta$, that is the smallest θ-closed subset of X containing A. The *θ-tightness* of X at $x \in X$ is $t_\theta(x, X) = \min\{\kappa :$ for every $A \subseteq X$ with $x \in cl_\theta(A)$ there exists $B \subseteq A$ such that $|B| \leq \kappa$ and $x \in cl_\theta(B)$; the *θ-tightness of* X is $t_\theta(X) = sup\{t_\theta(x, X) : x \in X\}$ [8]. We have that tightness and θ-tightness are independent (see Example 11 and Example 12 in [9]), but if X is a regular space then $t(X) = t_\theta(X)$. The θ-density of X is $d_\theta(X) = min\{\kappa : A \subseteq X, A$ is a dense subset of X and $|A| \leq \kappa\}$. We say that a subset A of X is *θ-dense* in X if $cl_\theta(A) = X$.

If X is a Hausdorff space, the *closed pseudocharacter of a point* x in X is $\psi_c(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a family of open neighborhoods of x and $\{x\}$ is the intersection of the closure of $\mathcal{U}\}$; the *closed pseudocharacter of* X is $\psi_c(X) = sup\{\psi_c(x, X) : x \in X\}$ (see [17] where it is called $S\psi(X)$). The *Urysohn pseudocharacter of* X, denoted by $U\psi(X)$, is the smallest cardinal κ such that for each point $x \in X$ there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that if $x \neq y$, then there exist $\alpha, \beta < \kappa$ such that $\overline{V(\alpha, x)} \cap \overline{V(\beta, y)} = \emptyset$ [18]; this cardinal function is defined only for Urysohn spaces. The *Urysohn-cellularity* of a space X is $Uc(X) = sup\{|\mathcal{V}| : \mathcal{V}$ is Urysohn-cellular$\}$ (a collection $\mathcal{V}$ of open subsets of X is called *Urysohn-cellular*, if O_1, O_2 in $\mathcal{V}$ and $O_1 \neq O_2$ implies $\overline{O_1} \cap \overline{O_2} = \emptyset$). Of course, $Uc(X) \leq c(X)$.

The *almost Lindelöf degree* of a subset Y of a space X is $aL(Y, X) = \min\{\kappa :$ for every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leq \kappa$ and $\bigcup\{\overline{V} : V \in \mathcal{V}'\} = Y\}$. The function $aL(X, X)$ is called the *almost Lindelöf degree* of X and denoted by $aL(X)$ (see [21] and [14]). The *almost Lindelöf degree of* X *with respect to closed subsets of* X is $aL_c(X) = \sup\{aL(C, X) : C \subseteq X$ *is closed*$\}$.

For a subset A of a space X we will denote by $[A]^{\leq\lambda}$ the family of all subsets of A of cardinality $\leq \lambda$.

Sapirovskii [16] proved that $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$, for a regular space X. Later Shu-Hao [17] proved that the previous inequality holds in the class of Hausdorff spaces by replacing the pseudocharacter with the closed pseudocharacter. In Section 2 we introduce the *θ-pseudocharacter* of a Urysohn space X, denoted by $\psi_\theta(X)$ and prove the following result:

- $|X| \leq \pi\chi(X)^{Uc(X)\psi_\theta(X)}$ for a Urysohn space X.

A space X is *n-Urysohn* [4] (resp. *n-Hausdorff* [3]), $n \in \omega$, if for every $x_1, x_2, ..., x_n \in X$ there exist open subsets $U_1, U_2, ..., U_n$ of X such that $x_1 \in U_1, x_2 \in U_2, ..., x_n \in U_n$ and $\bigcap_{i=1}^n \overline{U_i} = \emptyset$ (resp, $\bigcap_{i=1}^n U_i = \emptyset$). In [6] the authors generalized the Dissanayake and Willard's inequality: $|X| \leq 2^{aL_c(X)\chi(X)}$, for Hausdorff spaces X [21], in the class of n-Hausdorff spaces and de Groot's result: $|X| \leq 2^{hL(X)}$, for Hausdorff spaces [10], in the class of T_1 spaces. In particular, they used two new cardinal functions, denoted by $HW(X)$, $\psi w(X)$, to obtain the following results:

- If X is a T_1 n-Hausdorff ($n \in \omega$) space, then $|X| \leq HW(X)2^{aL_c(X)\chi(X)}$.
- If X is a T_1 space, then $|X| \leq HW(X)\psi w(X)^{haL(X)}$.

In Section 3 we introduce new cardinal functions, denoted by $UW(X)$, $\psi w_\theta(X)$, θ-$aL(X)$, $h\theta$-$aL(X)$, θ-$aL_c(X)$ and θ-$aL_\theta(X)$ such that $HW(X) \leq UW(X)$, $\psi w(X) \leq \psi w_\theta(X)$ and θ-$aL(X) \leq aL(X)$, restate Theorem 2.22 in [6] in the class of n-Urysohn spaces and give a variation of Theorem 2.23 in [6]. In particular, we prove the following results:

- If X is a T_1 n-Urysohn ($n \in \omega$) space, then $|X| \leq UW(X)2^{\theta\text{-}aL_\theta(X)\chi(X)}$.
- If X is a T_1 space then $|X| \leq UW(X)\psi w_\theta(X)^{h\theta\text{-}aL(X)}$.

In [5] the authors introduced the *Hausdorff point separating weight of a space* X denoted by $Hpsw(X)$ and proved a Hausdorff version of Charlesworth's inequality $|X| \leq psw(X)^{L(X)\psi(X)}$ [7]. In a similar way, in