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Exceptional cosmetic surgeries on homology spheres

Huygens C. Ravelomanana

Department of Mathematics, University of Georgia, Athens, GA 30602, USA

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The cosmetic surgery conjecture is a longstanding conjecture in 3-manifold theory. We present a theorem about exceptional cosmetic surgery for homology spheres. Along the way we prove that if the surgery is not a small Seifert $\mathbb{Z}/2\mathbb{Z}$-homology sphere or a toroidal irreducible non-Seifert surgery then there is at most one pair of exceptional truly cosmetic slope. We also prove that toroidal truly cosmetic surgeries on integer homology spheres must be integer homology spheres.

1. Introduction

In [15] we proved that for hyperbolic knots in S^3 the slope of *exceptional truly cosmetic surgeries* must be ± 1 and that the surgery must be irreducible toroidal and not Seifert fibred. As a consequence we showed that there are no truly cosmetic surgeries on alternating and arborescent knots in S^3. Here we study the problem for the case of integer homology spheres in general. Recall that a surgery on a hyperbolic knot is *exceptional* if it is not hyperbolic and that two surgeries on the same knot but with different slopes are called *cosmetic* if they are homeomorphic and are called *truly cosmetic* if the homeomorphism preserves orientation. The cosmetic surgery Conjecture [9, Conjecture (A) in problem 1.81] states that if the knot complement is boundary irreducible and irreducible then two surgeries on inequivalent slopes are never truly cosmetic. For more details about cosmetic surgeries we refer to [11], [1], [12], [13] and [15].

In this paper we study truly cosmetic surgeries along hyperbolic knots in homology spheres. We are concerned with the case where the two slopes are both exceptional slopes. We call such surgeries *exceptional truly cosmetic surgeries*. If K is a knot in an integer homology sphere Y, we denote $\mathcal{N}(K)$ a regular neighborhood of K, $Y_K := Y \setminus \mathrm{int}(\mathcal{N}(K))$ the exterior of K and $Y_K(r)$ Dehn surgery along K with slope r. When the manifold is a homology sphere we identify r with a rational number according to the standard

E-mail address: huygens@cirget.ca.

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meridian longitude basis where the longitude is the preferred longitude. The main result of this paper is the following.

Theorem 1.1. *Let K be a hyperbolic knot in a homology sphere Y. Let $Y_K(p/q)$ and $Y_K(p/q')$ be exceptional truly cosmetic surgeries, with $0 < p$ and $q < q'$. Then $Y_K(p/q)$ is either*

1. *a reducible manifold in which case $p = 1$ and $q' = q + 1$,*
2. *a toroidal Seifert fibred manifold in which case $p = 1$ and $q' = q + 1$,*
3. *a small Seifert manifold with infinite fundamental group in which case either*
   - *$p = 1$ and $|q - q'| \leq 8$.*
   - *or $p = 5$, $q' = q + 1$ and $q \equiv 2\,[\text{mod } 5]$.*
   - *or $p = 2$, and $q' = q + 2$ or $q' = q + 4$.*
4. *a toroidal irreducible non-Seifert fibred manifold in which case $p = 1$ and $|q' - q| \leq 3$.*

The following two corollaries are straightforward consequences of the theorem.

Corollary 1.2. *Toroidal truly cosmetic surgeries along hyperbolic knots in integer homology spheres yield integer homology spheres.*

Corollary 1.3. *For a hyperbolic knot in an homology sphere there is at most one pair of exceptional truly cosmetic slope which does not yield a $\mathbb{Z}/2\mathbb{Z}$-homology small Seifert surgery or a toroidal irreducible non-Seifert surgery.*

Notations If a torus T is a component of ∂M we denote $M(s, T)$ the Dehn filling of M with slope s along T, if ∂M has only one torus component we will simply write $M(s)$. In the case of surgery along a knot K is a 3-manifold Y we use the notation $Y_K(s)$ defined earlier.

Rational longitude Let K be a knot in a rational homology 3-sphere Y. The knot K has finite order in $H_1(Y, \mathbb{Z})$ so there is an integer n and a surface $\Sigma \subset Y$ such that $nK = \partial\Sigma$. The intersection of Σ with $\partial\mathcal{N}(K)$ is n-parallel copies of a curve λ_M. The isotopy class in $\partial\mathcal{N}(K)$ of this curve does not depend on the choice of the surface Σ. We call the corresponding slope the rational longitude of K and denote it by λ_M.

We will need the following lemma from [19].

Lemma 1.4 *([19]). Let s be a slope on ∂Y_K. There is a constant c_M such that*

$$|H_1(Y_K(s); \mathbb{Z})| = c_M\ \Delta(s, \lambda_M).$$

Here $\Delta(r, s)$ stands for the distance between two slopes r and s i.e. their minimal geometric intersection number.

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2. Proof of Theorem 1.1

Let us first recall a result of Boyer and Lines about the second derivative of the Alexander polynomial Δ''_K of a knot K.