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# Stability of intersections of graphs in the plane and the van Kampen obstruction

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ABSTRACT

A map $\varphi : K \to \mathbb{R}^2$ of a graph $K$ is *approximable by embeddings*, if for each $\varepsilon > 0$ there is an $\varepsilon$-close to $\varphi$ embedding $f : K \to \mathbb{R}^2$. Analogous notions were studied in computer science under the names of *cluster planarity* and *weak simplicity*. This short survey is intended not only for specialists in the area, but also for mathematicians from other areas.

We present criteria for approximability by embeddings (P. Minc, 1997, M. Skopenkov, 2003) and their algorithmic corollaries. We introduce *the van Kampen (or Hanani–Tutte) obstruction* for approximability by embeddings and discuss its completeness. We discuss analogous problems of moving graphs in the plane apart (cf. S. Spież and H. Toruńczyk, 1991) and finding closest embeddings (H. Edelsbrunner). We present higher dimensional generalizations, including completeness of the van Kampen obstruction and its algorithmic corollary (D. Repovš and A. Skopenkov, 1998).



## 1. Approximability by embeddings

### 1.1. Definition, examples and discussion

All the maps below are tacitly assumed to be continuous or piecewise-linear (PL).

A map $\varphi : K \to \mathbb{R}^2$ of a graph $K$ is **approximable by embeddings**, if for each $\varepsilon > 0$ there is an $\varepsilon$-close to $\varphi$ embedding $f : K \to \mathbb{R}^2$.

Even the cases when $\varphi$ is either a path or a cycle, i.e. either $K \cong I := [0, 1]$ or $K \cong S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, is interesting.

This notion appeared in studies of planarity of compacta and dynamical systems, see some illustrating examples in §1.2, §1.3. Related notions of *cluster planarity, weak simplicity, projected embedding, level-*

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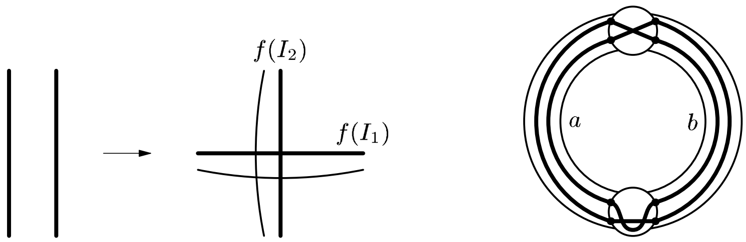


**Fig. 1.** Transversal intersection and the standard 2-winding are not approximable by embeddings.

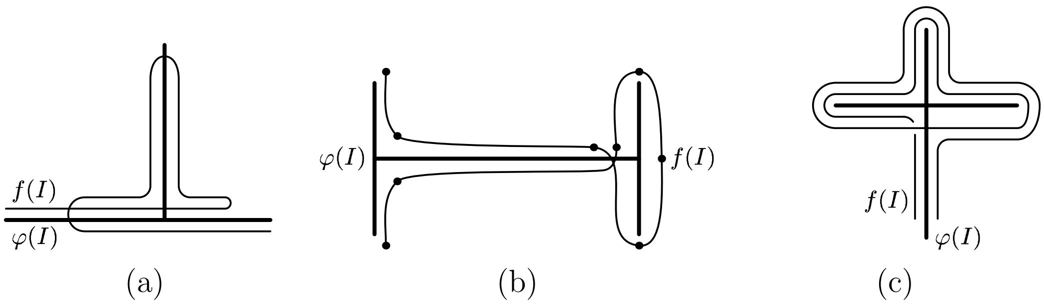


**Fig. 2.** Paths not approximable by embeddings and having no transversal intersections.

*planarity and monotone drawings* appeared in computer science and differential topology, see [4,14,15,25], [32, §3.5], [16, Theorem 1] and references therein.

*The following examples* (see also [26, Fig. 5]) show that this notion is interesting in itself. See elementary introduction in [9,20].

A *transversal self-intersection* of a PL map $\varphi : K \to \mathbb{R}^2$ is a pair of disjoint arcs $i, j \subset K$ such that $\varphi i$ and $\varphi j$ intersect transversally in the plane (see Fig. 1).

**Remark 1.1.**

(a) A map from a graph to a point in the plane is approximable by embeddings if and only if the graph is planar.

(b) Any map $f : I \to I \subset \mathbb{R}^2$ is approximable by embeddings. (Indeed, take the graph of $f$ in $I \times I \subset \mathbb{R}^2$ and compress it to the first factor.)

**Proposition 1.2.**

(a) *Any map $T \to I \subset \mathbb{R}^2$ is approximable by embeddings, where $T$ is triod (Sieklucki, 1969 [27]).*

(b) *The standard d-winding*

$$S^1 \to S^1 \subset \mathbb{R}^2, \quad (\cos\varphi, \sin\varphi) \mapsto (\cos d\varphi, \sin d\varphi)$$

*is approximable by embeddings if and only if $d \in \{-1, 0, 1\}$.*

*A map $S^1 \to S^1 \subset \mathbb{R}^2$ is approximable by embeddings if and only if its degree is in $\{-1, 0, 1\}$ (Sieklucki, 1969 [27]).*

(c) *An Euler path or an Euler cycle in the plane is approximable by embeddings if and only if it does not have transversal self-intersections (M. Skopenkov, 2003 [29]).*

(d) *There exists an algorithm of checking whether a given simplicial map is approximable by embeddings (A. Skopenkov, 1994, M. Skopenkov, 2003 [28,29]).*