Topology and its Applications 239 (2018) 1–13

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Topologies generated by porosity and maximal additive and multiplicative families for porouscontinuous functions

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A R T I C L E   I N F O

Article history:
Received 30 October 2017
Received in revised form 2 February 2018
Accepted 14 February 2018
Available online 19 February 2018

MSC:
54A10
54C30
26A15

Keywords:
Porosity
Strong porosity
Porouscontinuity
Maximal classes
Topology generated by porosity

A B S T R A C T

L. Zajíček in 1986 and V. Kelar in 1990 defined topologies p and s generated by the notions of porosity and strong porosity. Applying these notions, we described maximal additive classes for porouscontinuous functions $\mathcal{S}_0$, $\mathcal{M}_1$ and $\mathcal{P}_0$ defined by J. Borsík and J. Holos in 2014. Furthermore, we defined new family of topologies generated by porosity and strong porosity, which are used to studying maximal multiplicative classes for porouscontinuous functions $\mathcal{S}_0$ and $\mathcal{M}_1$. Some other properties of superporosity and strong superporosity are considered.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. For $f\colon Y \to Z$ and $A \subset Y$, by $f_{\restriction A}$ we mean the restriction of f to A. The symbol (X, ϱ) always stands for a metric space, $\mathrm{cl}(A)$ and $\mathrm{Int}(A)$ denote a closure and an interior of $A \subset X$. The aim of our paper is describing topologies on a metric space generated by porosity and strong porosity and their connections with the notions of porouscontinuity.

The open ball in (X, ϱ) with the center $x \in X$ and the radius R will be denoted by $B(x, R)$. Similarly, by $S(x, R)$ and $\overline{B}(x, R)$ we will denote a sphere and a closed ball with the center x and the radius R.

First, we recall the definitions of porosities. Let $M \subset X$, $x \in X$ and $R > 0$. Then, according to [8], we denote the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus M$

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https://doi.org/10.1016/j.topol.2018.02.010
0166-8641/

by $\gamma(x,R,M)$. The number $p(M,x) = 2\limsup_{R\to 0^+} \frac{\gamma(x,R,M)}{R}$ is called the porosity of M at x. Obviously, $p(M,x) = p(\mathrm{cl}(M),x)$ for $M \subset X$ and $x \in X$. Surprisingly, $p(M,x)$ may be equal to ∞ (for example if $X = (-\infty,-1] \cup \{0\} \cup \{\frac{1}{n!}: n \geq 1\}$ then $p((-\infty,-1],0) = \infty$). Nevertheless if $x \in M$ then $p(M,x) \leq 2$. Furthermore, it is easily seen that in a normed space one have $p(M,x) \leq 2$ and if $x \in M$ then $p(M,x) \leq 1$.

We say that the set M is porous at $x \in X$ if $p(M,x) > 0$. The set M is called porous if M is porous at each point $x \in M$. We say that the set M is strongly porous at x if $p(M,x) \geq 1$ and the set M is called strongly porous if M is strongly porous at each $x \in M$. Obviously every strongly porous set is porous and every porous set is nowhere dense. Moreover, none of reverse inclusions is true.

Remark 1.1. Let (X,ϱ) be a metric space, $A \subset M \subset X$ and $x \in X$. If $p(A,x) = 0$ then $p(M,x) = 0$.

In some applications we will use notions of porosities for subsets of $\mathbb{R}$. Due to L. Zajíček, J. Borsík and J. Holos [8,1] we give another definitions of porosities of subsets of the real line. Let the symbol $|J|$ stands for the length of an interval $J \subset \mathbb{R}$. For a set $A \subset \mathbb{R}$ and an interval $I \subset \mathbb{R}$ let $\Lambda(A,I)$ denote the length of the largest open subinterval of I having an empty intersection with A. Let $x \in \mathbb{R}$. Then, according to [1,8], the right-porosity of the set A at x is defined as

$$p^+(A,x) = \limsup_{h\to 0^+} \frac{\Lambda(A,(x,x+h))}{h},$$

the left-porosity of the set A at x is defined as

$$p^-(A,x) = \limsup_{h\to 0^+} \frac{\Lambda(A,(x-h,x))}{h},$$

and the porosity of A at x is defined as

$$p(A,x) = \max\left\{p^-(A,x), p^+(A,x)\right\}.$$

It is easy to see that for any $A \subset \mathbb{R}$ if $x \in A$ then definitions of $p(A,x)$ from [8] and from [1] are equivalent.

In [1,7] there are studied families of porouscontinuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Applying their ideas we convert this concept for real functions defined on a metric space.

Definition 1.2. Let (X,ϱ) be a metric space and $f\colon X \to \mathbb{R}$ and $x \in X$. The function f will be called:

- $\mathcal{P}_0$-continuous at x if there exists a set $A \subset X$ such that $x \in A$, $p(X \setminus A,x) > 0$ and $f_{\restriction A}$ is continuous at x;
- $\mathcal{S}_0$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset X$ such that $x \in A$, $p(X \setminus A,x) > 0$ and $f(A) \subset (f(x)-\varepsilon, f(x)+\varepsilon)$;
- $\mathcal{M}_1$-continuous at x if there exists a set $A \subset X$ such that $x \in A$, $p(X \setminus A,x) \geq 1$ and $f_{\restriction A}$ is continuous at x.

If f is $\mathcal{P}_0$-continuous, $\mathcal{S}_0$-continuous, $\mathcal{M}_1$-continuous at every point of X then we say that f is $\mathcal{P}_0$-continuous, $\mathcal{S}_0$-continuous, $\mathcal{M}_1$-continuous, respectively.

All of these functions are called porouscontinuous functions.

We will denote the class of $\mathcal{P}_0$-continuous, $\mathcal{S}_0$-continuous, $\mathcal{M}_1$-continuous functions by $\mathcal{P}_0$, $\mathcal{S}_0$, $\mathcal{M}_1$, respectively.

Obviously, if f is continuous at some x then f is porouscontinuous (in each sense) at x. Mainly, we will consider some special kind of metric spaces defined bellow.