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Products of H-separable spaces in the Laver model[☆]

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We prove that in the Laver model for the consistency of the Borel's conjecture, the product of any two H-separable spaces is M-separable.

1. Introduction

This paper is devoted to products of H-separable spaces. A topological space X is said [3] to be *H-separable*, if for every sequence $\langle D_n : n \in \omega\rangle$ of dense subsets of X, one can pick finite subsets $F_n \subset D_n$ so that every nonempty open set $O \subset X$ meets all but finitely many F_n's. If we only demand that $\bigcup_{n\in\omega} F_n$ is dense we get the definition of *M-separable* spaces introduced in [14]. It is obvious that second-countable spaces (even spaces with a countable π-base) are H-separable, and each H-separable space is M-separable. The main result of our paper is the following

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Theorem 1.1. *In the Laver model for the consistency of the Borel's conjecture, the product of any two countable H-separable spaces is M-separable.*

Consequently, the product of any two H-separable spaces is M-separable provided that it is hereditarily separable.

It worth mentioning here that by [12, Theorem 1.2] the equality $\mathfrak{b} = \mathfrak{c}$ which holds in the Laver model implies that the M-separability is not preserved by finite products of countable spaces in the strong sense.

Let us recall that a topological space X is said to have the *Menger property* (or, alternatively, is a *Menger space*) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X. This property was introduced by Hurewicz, and the current name (the Menger property) is used because Hurewicz proved in [7] that for metrizable spaces his property is equivalent to a certain property of a base considered by Menger in [10]. If in the definition above we additionally require that $\{n \in \omega : x \notin \cup \mathcal{V}_n\}$ is finite for each $x \in X$, then we obtain the definition of the *Hurewicz property* introduced in [8]. The original idea behind the Menger's property, as it is explicitly stated in the first paragraph of [10], was an application in dimension theory, one of the areas of interest of Mardešić. However, this paper concentrates on set-theoretic and combinatorial aspects of the property of Menger and its variations.

Theorem 1.1 is closely related to the main result of [13] asserting that in the Laver model the product of any two Hurewicz metrizable spaces has the Menger property. Let us note that our proof in [13] is conceptually different, even though both proofs are based on the same main technical lemma of [9]. Regarding the relation between Theorem 1.1 and the main result of [13], each of them implies a weak form of the other one via the following duality results: For a metrizable space X, $C_p(X)$ is M-separable (resp. H-separable) if and only if all finite powers of X are Menger (resp. Hurewicz), see [14, Theorem 35] and [3, Theorem 40], respectively. Thus Theorem 1.1 (combined with the well-known fact that $C_p(X)$ is hereditarily separable for metrizable separable spaces X) implies that in the Laver model, if all finite powers of metrizable separable spaces X_0, X_1 are Hurewicz, then $X_0 \times X_1$ is Menger. And vice versa: The main result of [13] implies that in the Laver model, the product of two H-separable spaces of the form $C_p(X)$ for a metrizable separable X, is M-separable.

The proof of Theorem 1.1, which is based on the analysis of names for reals in the style of [9], unfortunately seems to be rather tailored for the H-separability and we were not able to prove any analogous results even for small variations thereof. Recall from [6] that a space X is said to be *wH-separable* if for any *decreasing* sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, one can pick finite subsets $F_n \subset D_n$ such that for any non-empty open $U \subset X$ the set $\{n \in \omega : U \cap F_n \neq \emptyset\}$ is co-finite. It is clear that every H-separable space is wH-separable, and it seems to be unknown whether the converse is (at least consistently) true. Combining [6, Lemma 2.7(2) and Corollary 4.2] we obtain that every countable Fréchet–Urysohn space is wH-separable, and to our best knowledge it is open whether countable Fréchet–Urysohn spaces must be H-separable. The statement "finite products of countable Fréchet–Urysohn spaces are M-separable" is known to be independent from ZFC: It follows from the PFA by [2, Theorem 3.3], holds in the Cohen model by [2, Corollary 3.2], and fails under CH by [1, Theorem 2.24]. Moreover,[1] CH implies the existence of two countable Fréchet–Urysohn H-separable topological groups whose product is not M-separable, see [11, Corollary 6.2]. These results motivate the following

Question 1.2.

(1) Is it consistent that the product of two countable wH-separable spaces is M-separable? Does this statement hold in the Laver model?

[1] We do not know whether the spaces constructed in the proof of [1, Theorem 2.24] are H-separable.