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Monotone-light factorizations in coarse geometry

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A B S T R A C T

We introduce large scale analogues of topological monotone and light maps, which we call coarsely monotone and coarsely light maps respectively. We show that these two classes of maps constitute a factorization system on the coarse category. We also show how coarsely monotone maps arise from a reflection in a similar way to classically monotone maps, and prove that coarsely monotone maps are stable under those pullbacks which exist in the coarse category. For the case of maps between proper metric spaces, we exhibit some connections between the coarse and classical notions of monotone and light using the Higson corona. Finally, we look at some coarse properties which are preserved by coarsely light maps such as finite asymptotic dimension and exactness, and make some remarks on the situation for groups and group homomorphisms.

1. Introduction

Recall that continuous map from a compact Hausdorff space X to a compact Hausdorff space Y is called **monotone** if it is surjective and each of its fibres is connected, and is called **light** if each of its fibres is totally disconnected (see for example [25]). Eilenberg showed in [16] that every continuous map f between compact metric spaces factorizes as $f = me$, where m is light and e is monotone (in fact, the result holds more generally for compact Hausdorff spaces, see [7]). Moreover, this factorization satisfies a universal property, namely that for any commutative diagram

$$\begin{array}{ccccc} \bullet & \xrightarrow{e} & \bullet & \xrightarrow{m} & \bullet \\ {\scriptstyle u}\downarrow & & & & \downarrow{\scriptstyle v} \\ \bullet & \xrightarrow[e']{} & \bullet & \xrightarrow[m']{} & \bullet \end{array}$$

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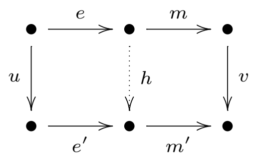
where the arrows are continuous maps and the objects are compact Hausdorff spaces, with e' monotone and m' light, there is a unique continuous map h making the diagram commute:



In the language of category theory, this is to say that the classes of monotone maps and light maps constitute a **factorization system** [17] on the category of compact Hausdorff spaces and continuous maps.

In coarse geometry one is interested in the large scale properties of metric (or more general) spaces, or in other words, properties of spaces "as viewed from a great distance" [20]. For example, the metric spaces $\mathbb{R}$ and $\mathbb{Z}$ are "coarsely equivalent", that is, isomorphic in the coarse category, despite being very different as topological spaces. The motivation to study large scale behaviour comes mostly from geometric group theory and index theory (see for example [18] and [22] respectively). Many classical topological notions have large scale analogues; for example, Gromov introduced the notion of asymptotic dimension in [18], which, when defined in terms of covers of a space, is clearly analogous to the covering dimension of a topological space. It was later shown that for proper metric spaces, the asymptotic dimension coincides with the (topological) covering dimension of the Higson corona (a topological space that captures large scale behaviour of a metric space) whenever the former is finite [10]. This gives another connection between the large scale and topological notions of dimension.

In this paper, we introduce large scale analogues of the topological monotone and light maps mentioned above, to which we give the names coarsely monotone and coarsely light maps respectively. These classes of maps will turn out to constitute a factorization system on the coarse category (defined in the next section). A large part of the paper is devoted to making some connections between the topological and large scale notions of monotone and light. We do so in two ways. Firstly, we examine these classes of maps from a categorical perspective inspired by the results in [7]. Secondly, we make some connections using the Higson corona in the case when the large scale spaces involved are proper metric spaces. Coarsely light maps generalize both coarse embeddings and coarsely n-to-1 maps; we prove that coarsely light maps preserve certain coarse properties such as finite asymptotic dimension and Yu's Property A in a similar way to these classes of maps. In the final section of the paper, we make some remarks on coarsely monotone and light maps between groups.

The main goal of this paper is to introduce two interesting classes of maps between large scale spaces and study some of their properties. Along the way, however, we also investigate some of the structure of the coarse category and apply some basic categorical arguments to large scale spaces and maps between them. It would be interesting to see what other categorical notions turn out to be useful in the study of large scale spaces.

2. Preliminaries

We recall some basic terminology from [12]. Let X be a set. Recall that the **star** $\mathrm{st}(B,\mathcal{U})$ of a subset B of X with respect to a family $\mathcal{U}$ of subsets of X is the union of those elements of $\mathcal{U}$ that intersect B. More generally, for two families $\mathcal{B}$ and $\mathcal{U}$ of subsets of X, $\mathrm{st}(\mathcal{B},\mathcal{U})$ is the family $\{\mathrm{st}(B,\mathcal{U}) \mid B \in \mathcal{B}\}$.

Definition 2.1. A **large scale structure** $\mathcal{L}$ on a set X is a nonempty set of families $\mathcal{B}$ of subsets of X (which we call the **uniformly bounded families** in X) satisfying the following conditions: