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Asymptotic dimension of coarse spaces via maps to simplicial complexes ☆

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ABSTRACT

It is well-known that a paracompact space X is of covering dimension at most n if and only if any map $f\colon X \to K$ from X to a simplicial complex K can be pushed into its n-skeleton $K^{(n)}$. We use the same idea to characterize asymptotic dimension in the coarse category of arbitrary coarse spaces. Continuity of the map f is replaced by variation of f on elements of a uniformly bounded cover. In the same way, one can generalize Property A of G. Yu to arbitrary coarse spaces.

1. Introduction

It is well-known (see [3]) that the covering dimension $\dim(X)$ of a paracompact space can be defined as the smallest integer n with the property that any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & K^{(n)} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & K \end{array}$$

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has a filler h

$$\begin{array}{ccc} A & \xrightarrow{g} & K^{(n)} \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & K \end{array}$$

Here A is any closed subset of X, K is any simplicial complex with the metric topology, $K^{(n)}$ is the n-skeleton of K, and $i\colon A \to X$, $i\colon K^{(n)} \to K$ are inclusions. By saying h is a **filler** we mean $h|A = g$ and, since we cannot insist on $i \circ h = f$, we require $h(x) \in \Delta$ whenever $f(x) \in \Delta$ for any simplex Δ of K.

In [1], a generalization of the above result was announced for the coarse category of metric spaces. However, Kevin Zhang, a Ph.D. student in Fudan University of China, found a gap in that paper. Therefore, the goal of the present paper is not only to provide a proof but to generalize the result even further, namely to the category of arbitrary coarse spaces. This is done by demonstrating existence of useful partitions of unity for point-finite covers of coarse spaces (see Section 3).

In our work we will not use the original description of the coarse category of J. Roe [6]. Instead, we will rely on the alternative description provided in [4] that is more useful in the context of asymptotic dimension.

The first issue is to find the analog of continuous maps $f\colon X \to K$ from X to a simplicial complex K.

As seen in [3] the optimal way to define paracompact spaces X is as follows: for each open cover $\mathcal{U}$ of X there is a simplicial complex K and a continuous map $f\colon X \to K$ such that the family $\{f^{-1}(st(v))\}_{v\in K^{(0)}}$ refines $\mathcal{U}$, where the **star** $st(v)$ of vertex v is the union of interiors of all simplices of K containing v.

In [1] the continuous functions $f\colon X \to K$ were replaced by (λ, C)-Lipschitz functions and the analog of paracompact spaces in coarse geometry was defined as follows:

Definition 1.1. [1] A metric space X is **large scale paracompact** (ls-paracompact for short) if for each uniformly bounded cover $\mathcal{U}$ of X and for all $\lambda, C > 0$ there is a (λ, C)-Lipschitz function $f\colon X \to K$ such that $\mathcal{V} := \{f^{-1}(st(v))\}_{v\in K^{(0)}}$ is uniformly bounded and $\mathcal{U}$ refines $\mathcal{V}$.

To simplify Definition 1.1 the following concept was introduced:

Definition 1.2. [1] Given $\delta > 0$ and a simplicial complex K, a function $f\colon X \to K$ is called a **δ-partition of unity** if it is (δ, δ)-Lipschitz, $\mathcal{V} := \{f^{-1}(st(v))\}_{v\in K^{(0)}}$ is uniformly bounded, and the Lebesgue number of $\mathcal{V}$ is at least $\frac{1}{\delta}$.

For arbitrary coarse spaces we need different but related concepts.

Definition 1.3. Given a cover $\mathcal{U}$ of a set X and given a function $f\colon X \to M$ from X to a metric space M, the **$\mathcal{U}$-variation** $var_{\mathcal{U}}(f)$ of f is the supremum of $d(f(x), f(y))$, where $\{x, y\}$ is contained in a single element of $\mathcal{U}$.

Definition 1.4. Given a cover $\mathcal{U}$ of a coarse space X, given $\epsilon > 0$, and given a partition of unity $f\colon X \to K$, we say f is a **$(\mathcal{U}, \epsilon)$-partition of unity** if the following conditions are satisfied:
a. $var_{\mathcal{U}}(f) < \epsilon$,
b. for every $U \in \mathcal{U}$ there is $v \in K^{(0)}$ such that $f(y) \in st(v)$ for all $y \in U$. In other words, point-inverses under f of stars of vertices of K coarsen $\mathcal{U}$,
c. point-inverses under f of stars of vertices of K form a uniformly bounded cover of X.

We are grateful to Kevin Zhang for pointing out a gap in the paper [1].