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Virtual Special Issue – In memory of Professor Sibe Mardešić

The shape of the Julia set of an expanding rational map

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A B S T R A C T

In memory of Sibe Mardešić, our friend.

Sibe Mardešić has enriched algebraic topology developing shape and strong shape theories with important constructions and theorems.

This paper relates computational topology to shape theory. We have developed some algorithms and implementations that under some conditions give a shape resolution of some Julia sets.

When a semi-flow is induced by a rational map g of degree d defined on the Riemann sphere, one has the associated Julia set $J(g)$. The main objective of this paper is to give a computational procedure to study the shape of the compact metric space $J(g)$.

Our main contribution is to provide an inverse system of cubic complexes approaching $J(g)$ by using implemented algorithms based in the notion of spherical multiplier. This inverse system of cubical complexes is used to: (i) obtain nice global visualizations of the fractal structure of the Julia set $J(g)$; (ii) determine the shape of the compact metric space $J(g)$.

These techniques also give the possibility of applying overlay theory (introduced by R. Fox and developed among others by S. Mardešić) to study the symmetry properties of the fractal geometry of the Julia set $J(g)$.

0. Introduction

The main aim of this paper is to study the shape of the Julia set of a rational map by means of topological, geometrical and computational tools.

From the geometrical point of view, we consider three models for the Riemann sphere: The complex projective line $\mathbf{P}^1(\mathbb{C})$, the Alexandroff compactification of the complex space $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the unit 2-sphere S^2. In this paper, the model $\hat{\mathbb{C}}$ is used to represent the domain of a rational map when it is given as a quotient of two univariate polynomials. Using the model $\mathbf{P}^1(\mathbb{C})$ we can work with normalized homogeneous

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coordinates and a rational map can be represented by a pair of homogeneous bivariate polynomials of the same degree (this technique permits us the iteration of a rational map without overflow and underflow problems). Finally, the Riemannian structure of S^2 is used to define the spherical multiplier map and to analyze the convergence of forward orbits to attracting cycles.

We also consider some topological and combinatorial techniques: Let **Top** denote the category of topological spaces and its localization Ho(**Top**) which is the category obtained by inverting all the weak equivalences. Let pro-$\mathcal{C}$ denote the category of inverse systems (or projective systems) of a category $\mathcal{C}$. In particular, we consider the categories pro-**Top** and pro-Ho(**Top**). Given a manifold M and a compact subspace $J \subset M$, the *shape* of J can be defined as the isomorphism class in pro-Ho(**Top**) of the inverse system of the neighborhoods at J induced by the topology of M. If we choose two different bases of neighborhoods at J, the two inverse systems of basic neighborhoods determine the same shape. It is also interesting to remark that the shape only depends on J and it does not depend on the 'ambient manifold' M. In this paper, we construct some inverse systems (resolutions using the terminology of shape theory, see [20]) for the Julia set $J(g) \subset S^2$. In this case, we obtain an inverse system of cubic complexes. This permits us the calculus of shape invariants by computing the standard homotopy invariants of each cubic complex in the inverse system. For an introduction with more details about shape theory we refer the reader to Edwards–Hasting's monograph [5], Dydak–Segal's monograph [3], Mardešić–Segal's book [20], Lisica–Mardešić's paper [17] and Porter's paper [24].

The main contribution of this paper is the computational construction of the shape resolutions given in Theorem 10 by using the spherical multiplier introduced in section 3 and the formulas given in sections 1 and 2. We have developed and implemented an algorithm that permits us to compute a finite number of cubic complexes of these resolutions. This can be use to obtain global visualizations of the Julia set of an expanding rational map or to compute homotopical or homological shape invariants. We have also compared the global visualization algorithm based on spherical multiplier to other implemented algorithms, previously developed by the authors, based either on the study of repelling cycles or on the topological frontier of the basin of a non-repelling cycle.

We would like to point out that most of the algorithms and computer applications in the literature only visualize the part of the Julia set corresponding to a rectangular region. However, our algorithms are designed to obtain a global graphic representation of the Julia set on the unit 2-sphere.

In this paper we have tried to establish a bridge between computational topology and shape and strong shape theory. The algorithms developed in this work can be extended in order to compute (direct) inverse systems of (co)homology groups which are shape invariants whose properties have been studied among others by S. Mardešić [19], [20]. We also want to mention that we are developing a project whose aim is to analyze the symmetry of some Julia sets which is based on the results of Mardešić–Matijević [21] about the classification of overlays.

The authors of this contribution want to give this posthumous acknowledgment to Professor Sibe Mardešić. We recall his visits to Spain and his wonderful lectures about shape and strong shape theory (during his stay in La Rioja (Spain), Sibe used to speak to us in Spanish; he had learned Spanish when he was a child in Chile). His mathematical ideas and enthusiasm for work have influenced our topological research lines very much.

1. Preliminaries and notation

The main objective of this work is to construct a shape resolution of a Julia set based on the notion of spherical multiplier of an analytic function defined on the Riemann sphere. The design of this algorithm requires the use of many notions and different representations of points, spaces and maps. Here we give a list of some of these concepts: a cyclic point of a semi-flow, an atlas of the 2-sphere, normalized homogeneous coordinates, homogeneous representation of a rational map, the tangent map of an analytic map, the Rie-