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Degree of homogeneity on suspensions of manifolds

Shijie Gu

Department of Mathematical Sciences, University of Wisconsin, Milwaukee, WI, 53211, United States

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Let M^n be a closed connected finite dimensional manifold. The suspension ΣM^n of M^n is homogeneous if and only if M^n is homeomorphic to a topological sphere S^n. Furthermore, ΣM^n is $\frac{1}{2}$-homogeneous if and only if M^n is not homeomorphic to S^n.

1. Introduction

An *orbit* of a space X is the group action of $\mathscr{H}(X)$ at a point $x \in X$, where $\mathscr{H}(X)$ is the group of homeomorphisms of X onto itself. A space X is said to be *homogeneous* or 1-*homogeneous* if for any two points $x, y \in X$, there is a homeomorphism of X onto itself taking x to y. Given a positive integer n, a space is said to be $\frac{1}{n}$-*homogeneous* provided that X has exactly n orbits. There have been increasing studies on this topic, especially, the $\frac{1}{2}$-homogeneity. Interested readers might find extensive discussions regarding continua, ANR-spaces, hyperspaces, cones and suspensions in [1–7].

This paper concerns a question posed by M. de J. López et al. [1, Question 4.6.14]: For which n-manifold M^n it is true that the suspension ΣM^n is $\frac{1}{2}$-homogeneous? The following are the main results of this paper.

Theorem 1.1. *Let M^n be a closed connected n-dimensional manifold. The suspension ΣM^n of M^n is* 1-*homogeneous if and only if M^n is homeomorphic to a topological sphere S^n.*

E-mail address: shijiegu@uwm.edu.

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Corollary 1.2. *Let M^n be a closed connected n-dimensional manifold. The suspension ΣM^n of M^n is $\frac{1}{2}$-homogeneous if and only if M^n is not homeomorphic to S^n.*

Succinctly speaking, the above results are obtained by combining the work on a particular stratified space from [11] with the generalized Poincaré conjecture.

2. Definitions and notations

Definition 2.1. A metrizable space X is said to be an n-dimensional *locally cone-like space* ($n \geq 2$) if each point $x \in X$ has a neighborhood U_x such that U_x is homeomorphic to an open cone $C(S_x)$ over a connected and compact $(n-1)$-dimensional locally cone-like space S_x with x denotes the vertex of the cone $C(S_x)$. We assume the empty set to be the unique (-1)-dimensional locally cone-like space. A singleton is a 0-dimensional locally cone-like space. A closed interval and circle S^1 are 1-dimensional locally cone-like spaces.

Every locally cone-like space X of dimension n has a natural topological stratification $X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that $X_i - X_{i-1}$ is an i-dimensional topological manifold. We refer readers to [10] for more details about stratified spaces.

Remark 2.1. By the fibration theories [8,9], Alexandrov spaces are locally cone-like spaces. Equivalently, such spaces are also called spaces with multiple conic singularities (MCS).

Definition 2.2. A locally compact separable metric space X is called a *homology n-manifold* if for each $x \in X$,

$$H_*(X, X - x) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - \{\text{origin}\}),$$

where $H_*(\cdot, \cdot)$ denotes singular relative homology with integer coefficients.

The symbol $S(X)$ is used to denote the singular set or nonmanifold set of a homology manifold X; for the dimension of $S(X)$ we use $\dim S(X)$.

Definition 2.3. A *homology sphere* is an n-manifold X having the homology groups of an n-sphere.

3. Proof of Theorem 1.1

The following two propositions discovered in [11] present important information about locally cone-like homology manifolds.

Proposition 3.1. *([11, Prop. 2.2]) Let X be a locally cone-like space of dimension 3. If X is a homology manifold, then it is a topological manifold.*

Proposition 3.2. *([11, Prop. 2.3]) Let X be a locally cone-like space of dimension $4 \leq n < \infty$. If X is a homology manifold, then $\dim S(X) \leq 1$.*

Combining the preceding propositions, we have

Lemma 3.1. *Suppose X is a finite dimensional locally cone-like space. If X is a connected and homogeneous homology manifold, then X is a topological manifold.*