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On metric spaces where continuous real valued functions are uniformly continuous and related notions

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ABSTRACT

Given a metric space $\mathbf{X} = (X, d)$ we show in **ZF** that:

(a) The following are equivalent:
(i) For every two closed and disjoint subsets A, B of $\mathbf{X}$, $d(A, B) > 0$.
(ii) Every countable open cover of $\mathbf{X}$ has a Lebesgue number.
(iii) Every real valued continuous function on $\mathbf{X}$ is uniformly continuous.
(iv) For every countable (resp. finite, binary) open cover $\mathcal{U}$ of $\mathbf{X}$, there exists a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$, $\{x, y\} \subseteq U$ for some $U \in \mathcal{U}$.

(b) If $\mathbf{X}$ is connected then: $\mathbf{X}$ is countably compact iff every open cover of $\mathbf{X}$ has a Lebesgue number iff for every two closed and disjoint subsets A, B of $\mathbf{X}$, $d(A, B) > 0$.

1. Notation and terminology

Let $\mathbf{X} = (X, d)$ be a metric space, $x \in X$ and $\varepsilon > 0$. $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ denotes the open ball in $\mathbf{X}$ with center x and radius ε. Given $B \subseteq X$, $\delta(B) = \sup\{d(x, y) : x, y \in B\} \in \mathbb{R}_+ \cup \{+\infty\}$ will denote the *diameter* of B.

Let $\mathcal{U}$ be an open cover of $\mathbf{X}$. $\mathcal{U}$ is said to be *binary* iff it contains two elements. We say that $\mathcal{U}$ has a *Lebesgue number* $\delta > 0$ iff for every $A \subseteq X$ with $\delta(A) < \delta$ there exists $U \in \mathcal{U}$ with $A \subseteq U$. An open refinement of $\mathcal{U}$ is a new collection of open sets such that each set in the new collection is contained in some member of $\mathcal{U}$. $\mathcal{U}$ is said to be *locally finite* if each point of X has a neighborhood intersecting a finite number of elements of $\mathcal{U}$.

$\mathbf{X}$ is said to be *Lebesgue (*resp. *countably Lebesgue)* iff every open cover $\mathcal{U}$ (resp. countable open cover) of $\mathbf{X}$ has a Lebesgue number.

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X is said to be *normal* iff the distance of every two disjoint, non-empty closed subsets of **X** is strictly positive.

X is said to be *totally bounded* iff for every $\varepsilon > 0$, **X** can be covered by finitely many open balls of radius ε.

A family $\mathcal{G}$ of closed subsets of **X** is called *uniformly discrete* iff there is $\delta > 0$ such that for every $F, G \in \mathcal{G}$, $F \neq G$, $d(F, G) > \delta$.

Below we list the two weak forms of the axiom of choice we shall use in this paper.

- **CAC** (Form 8 in [4]), Countable Axiom of Choice: For every countable family $\mathcal{A}$ of non-empty sets there exists a function f such that for all $x \in \mathcal{A}$, $f(x) \in x$.
- $\mathbf{PKW}(\aleph_0, \geq 2, \infty)$ (Form 167 in [4]), Partial Kinna–Wagner Principle: Every disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ such that for all $i \in \omega, |A_i| \geq 2$ has a partial Kinna–Wagner choice, i.e., there exists an infinite subfamily $\mathcal{B} = \{A_{k_i} : i \in \omega\}$ of $\mathcal{A}$ and a family $\mathcal{F} = \{F_i : i \in \omega\}$ of non-empty sets such that for all $i \in \omega$, $F_i \subsetneq A_{k_i}$.

2. Introduction

In this paper we continue with the study of the notions of countably Lebesgue, *UC* and normal metric spaces initiated in [7]. The intended context for reasoning will be **ZF** unless otherwise noted. In order to stress the fact that a result is proved in **ZF** (resp. **ZF** + **CAC**) we shall write in the beginning of the statements of the theorems (**ZF**) (resp. (**ZF**+**CAC**)).

The *UC* spaces were introduced by Atsuji in [1]. In [6], [3], [2] they are called Atsuji spaces. A great deal of information regarding these spaces can be found in the expository paper [6] by S. Kundu, T. Jain.

A. Monteiro and M. Peixoto in [10] proved the following theorem.

Theorem 1. *[10]* (**ZF**) *Let* $\mathbf{X} = (X, d)$ *be a metric space. The following are equivalent:*
(L): Every binary open cover of **X** *has a Lebesgue number;*
(F): The distance of every two disjoint, non-empty closed subsets of **X** *is strictly positive;*
(L′): Every finite open cover of **X** *has a Lebesgue number.*

The same authors have also proved, in (**ZF**+**CAC**), in that a metric space is *UC* iff it is Lebesgue (in [10], Lemma 3, p. 108 the authors use **CAC** to fix a sequence $(x_n)_{n\in\mathbb{N}}$ of points of the metric space **E** such that for all $n \in \mathbb{N}$, $B(x_n, 1/n)$ is included in no member of the open cover $\mathcal{C}$). Subsequently, S. Mrówka in [11] called a metric space satisfying property (L) of Theorem 1 normal, and proved in (**ZF**+**CAC**), that a metric space is *UC* iff it is normal. The notion of countably Lebesgue metric space was introduced in [7] and has been established, in (**ZF**+**CAC**) again, that a metric space is Lebesgue iff it is countably Lebesgue. Hence, in (**ZF**+**CAC**),

- Lebesgue $\leftrightarrow$ countably Lebesgue $\leftrightarrow$ *UC* $\leftrightarrow$ normal.

In (**ZF**) however, the implication “countably Lebesgue $\rightarrow$ Lebesgue” is no more valid, i.e., the negation of the statement: Every countably Lebesgue metric space is Lebesgue is relatively consistent with **ZF**. In [7] it has been shown that “countably Lebesgue $\rightarrow$ Lebesgue” $\rightarrow \mathbf{PKW}(\aleph_0, \geq 2, \infty)$. Since there are **ZF** models, see e.g., Cohen’s Second Model $\mathcal{M}7$ in [4], satisfying the negation of $\mathbf{PKW}(\aleph_0, \geq 2, \infty)$, it follows that in $\mathcal{M}7$ there exist countably Lebesgue, non-Lebesgue metric spaces.

The following **ZF** implications have been established in [7].