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Ample continua in Cartesian products of continua

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We show that the Cartesian product of the arc and a solenoid has the fupcon property, therefore answering a question raised by Illanes. This combined with Illanes' result implies that the product of a Knaster continuum and a solenoid has the fupcon property, therefore answering a question raised by Bellamy and Łysko in the affirmative. Finally, we show that a product of two Smith's nonmetric pseudo-arcs has the fupcon property.

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1. Introduction

The present paper is concerned with the property of having arbitrarily small open neighborhoods for continua in Cartesian products of continua; i.e. given a continuum $M \subseteq X \times Y$ we are interested if

(*) *for every open neighborhood U of M there exists an open and connected set V such that $M \subseteq V \subseteq U$.*

The property (*) is closely related to the property of being an ample[1] continuum in the product. Recall that M is *ample* in $X \times Y$ provided that for each open subset $U \subseteq X \times Y$ such that $M \subseteq U$, there exists a subcontinuum L of $X \times Y$ such that $M \subseteq \text{int}_{X\times Y}(L) \subseteq L \subseteq U$. In fact, according to [1], the two

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[1] The notion of an ample continuum was introduced by Prajs and Whittington in [10].

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properties are equivalent in the class of Kelley continua. Motivation for the study of ample continua comes from fact that in the hyperspace $C(X \times Y)$ of subcontinua of $X \times Y$ ample continua are the points where $C(X \times Y)$ is locally connected. In this context in [1] Bellamy and Łysko studied the *fupcon*[2] property of Cartesian products. The product of continua $X \times Y$ has the fupcon property if whenever $M \subseteq X \times Y$ is a continuum with full projections onto coordinate spaces (i.e. $\pi_X(M) = X$ and $\pi_Y(M) = Y$) then M has the property (*), and the notion naturally generalizes to Cartesian products of more than two continua. Bellamy and Łysko showed that arbitrary Cartesian products of Knaster continua and arbitrary Cartesian products of pseudo-arcs have the fupcon property. Furthermore, the property (*) for subcontinua of such products is in fact equivalent to the property of having full projections onto all coordinate spaces. The authors also showed that the diagonal in a Cartesian square G of a compact and connected topological group has the property (*) if and only if G is locally connected, and therefore if G is a solenoid then $G \times G$ does not have the fupcon property. Important related results on ample diagonals can be found in the recent work of Prajs [9]. Motivated by the aforementioned results, Bellamy and Łysko raised the following question.

Question 1 *(Bellamy & Łysko [1]). Let K be a Knaster continuum and S be a solenoid. Does $K \times S$ have the fupcon property?*

A partial step towards a solution to the above problem was achieved by Illanes, who showed the following.

Theorem A *(Illanes [7]). Let X be a continuum such that $X \times [0, 1]$ has the fupcon property. Then for each Knaster continuum K, $X \times K$ has the fupcon property.*

Consequently, Question 1 was reduced to the following, potentially simpler problem.

Question 2 *(Illanes [7]). Let S be a solenoid. Does $[0, 1] \times S$ have the fupcon property?*

We answer this question in the affirmative, and in turn obtain positive answer to Question 1.

Theorem 1.1. *Let S be a solenoid. Then $[0, 1] \times S$ has the fupcon property.*

Theorem 1.2. *Let S be a solenoid and K be a Knaster continuum. Then $K \times S$ has the fupcon property.*

In 1985 M. Smith [11] constructed a nonmetric pseudo-arc $\mathcal{M}$; i.e. a Hausdorff chainable, homogeneous, hereditary equivalent and hereditary indecomposable continuum. This continuum has been recently used by the first and third author to provide a new counterexample to Wood's Conjecture in the isometric theory of Banach spaces [2]. Relying on the result of Bellamy and Łysko that products of metric pseudo-arcs have the fupcon property, we shall show that their result holds also for products of $\mathcal{M}$.

Theorem 1.3. *Let $\mathcal{M}$ be Smith's nonmetric pseudo-arc. Any Cartesian power of $\mathcal{M}$ has the fupcon property.*

Earlier, Lewis showed [8] that for any 1-dimensional continuum X there exists a 1-dimensional continuum X_P that admits a continuous decomposition into pseudo-arcs, and whose decomposition space is homeomorphic to X. Recently, Boroński and Smith [3] extended Lewis' result to continuous curves of Smith's nonmetric pseudo-arc. In particular, given any metric 1-dimensional continuum X there exists a continuum $X_{\mathcal{M}}$ that admits a continuous decomposition into nonmetric pseudo-arcs, and whose decomposition space is homeomorphic to X. $X_{\mathcal{M}}$ can be seen as "X of nonmetric pseudoarcs". Here we observe that using the method of proof of Theorem 1.3 one obtains the following generalization.

[2] The abbreviation *fupcon* stands for *full projections imply connected open neighborhoods*. It was introduced by Illanes in [7].

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