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Commutator subgroups of welded braid groups ☆

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ABSTRACT

Let WB_n be the welded (or loop) braid group on n strands, $n \geq 3$. We investigate commutator subgroup of WB_n, WB_n'. We prove that WB_n' is finitely generated and Hopfian. We show that WB_n' is perfect if and only if $n \geq 5$. We also compute finite presentation for FWB_n', the commutator subgroup of the flat welded braid group FWB_n. Along the way, we investigate adorability of these groups.

1. Introduction

Welded braid groups are certain extensions of the classical braid groups. These groups have appeared in several contexts in the literature and often with different names, e.g. [4], [5], [7], [9] [10]. They are also known as *loop braid groups* or *permutation braid groups* or *symmetric automorphisms of free groups*. We refer to the recent survey article by Damiani [6] for further details on different definitions and applications of these groups.

In this paper, we investigate the commutator subgroups of the welded braid groups. The commutator B_n' of the classical braid group B_n is well studied. Gorin and Lin [8] obtained a finite presentation of B_n'. Simpler presentation of B_n' was obtained by Savushkina [20]. Several authors have investigated the larger

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class of spherical Artin groups, e.g. [22], [16], [17]. In this context, it is a natural question to investigate structures of commutator subgroups of other classes of generalized braid groups.

Let WB_n denote the welded braid group of n strands. We investigate the commutator subgroup of WB_n. Recall that a group G is called *perfect* if it is equal to its commutator subgroup. We prove the following:

Theorem 1.1. *Let WB_n' denote the commutator subgroup of the welded braid group WB_n.*

(i) *WB_n' is a finitely generated group for all $n \geq 3$. For $n \geq 7$, the rank of WB_n' is at most $1 + 2(n - 3)$, and for $3 \leq n \leq 6$, the rank is at most $4 + 2(n - 3)$.*
(ii) *For $n \geq 5$, WB_n' is perfect.*

It is known that $WB_2 = F_2 \rtimes S_2$. So, the commutator WB_2' is infinitely generated.

Recall that a group G is called *Hopfian* if every epimorphism $G \to G$ is an isomorphism. In general, being Hopfian is not a subgroup-closed group property. Using the above theorem, we have the following.

Corollary 1.2. *For any $n \geq 3$, WB_n' is Hopfian.*

Another consequence of the above theorem is the following.

Corollary 1.3. *For a free group F_k, the image of any nontrivial homomorphism $\phi : WB_n \to F_k$ is infinite cyclic.*

The Reidemeister–Schreier method is a standard technique to obtain presentations of subgroups; for details see [15]. This method was used to obtain presentations of certain classes of Artin groups in [14], [12], [16]. We shall use this method to compute a presentation for WB_n'. We shall first find out a presentation using Reidemeister–Schreier method, and then using Tietze transformations, will eliminate redundant generators to obtain a finite generating set.

Adorability of welded braid groups. Motivated by the covering theory of aspherical 3-manifolds, Roushon defined the notion of an adorable group: a group G is called *adorable* if $G^i/G^{i+1} = 1$ for some i, where $G^i = [G^{i-1}, G^{i-1}]$ and $G^0 = G$ are the terms in the derived series of G. The smallest i for which the above property holds, is called the *degree of adorability* of G. For more details on adorable groups, see [18,19]. Applying Roushon's results with part (ii) of the above theorem, we have the following.

Corollary 1.4. *For $n \geq 5$, WB_n is adorable of degree 1, and for $n = 3, 4$, WB_n is not adorable.*

Therefore, by Theorem 1.1 and Corollary 1.4 we immediately have the following.

Corollary 1.5. *The group WB_n' is perfect if and only if $n \geq 5$.*

This generalizes the fact that B_n is adorable of degree 1 for $n \geq 5$. It is easy to see that if $f : G \to H$ be a surjective homomorphism with G adorable, then H is also adorable and $doa(H) \leq doa(G)$, where $doa(G)$ denotes degree of adorability, see [19, Lemma 1.1]. It follows from [2, Proposition 8] that the commutator subgroup VB_n' of the virtual braid group VB_n is perfect for $n \geq 5$. Thus, for $n \geq 5$, VB_n is adorable of degree 1. The welded braid groups being quotients of these groups, are also adorable with degree ≤ 1. This gives a proof of the fact that WB_n' is perfect for $n \geq 5$ even without using the presentation of WB_n'. Using the presentation, we give a direct proof of this fact.

After finishing this article, we have come to know about the recent work of Zaremsky [21] that implies the finite presentability of WB_n' for $n \geq 4$, see [21, Theorem B]. The finite generation of this group for $n \geq 3$ is