Topology and its Applications 237 (2018) 26–36

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The probabilistic powerdomain from a topological viewpoint ☆

Zhenchao Lyu, Hui Kou *

Department of Mathematics, Sichuan University, Chengdu 610064, China

ARTICLE INFO

Article history:
Received 23 April 2017
Received in revised form 15 January 2018
Accepted 17 January 2018
Available online 31 January 2018

MSC:
54A99
06B30
68Q55

Keywords:
c-space
Locally finitary compact
Coherent
Restriction and corestriction
Probabilistic powerdomain

ABSTRACT

We prove that the probabilistic powerdomain of a coherent locally finitary compact T_0 space is coherent quasicontinuous. As a result, we obtain a novel proof of Larrecq's and Jung's result in 2014. The main tool for our proof is the weak topology on the probabilistic powerdomain. In addition, we show that a dcpo L is continuous (resp., quasicontinuous, coherent quasicontinuous, meet-continuous) if the probabilistic powerdomain $\mathcal{V}(L)$ over L is continuous (resp., quasicontinuous, Lawson compact quasicontinuous, meet-continuous), which confirms a conjecture of Jones in her doctoral thesis.

1. Introduction

In 1989, Claire Jones and Gordon Plotkin introduced the probabilistic powerdomain of a topological space, and proved that the probabilistic powerdomain of a continuous domain equipped with the Scott topology is still a continuous domain. Over the past two decades, many scholars have done a great deal of work about probabilistic powerdomain, i.e., [3,6,10–13,8,14,9].

Up to now, there are four full subcategories of dcpos that are known to be closed under the probabilistic powerdomain construction: the category of dcpos, that of continuous domains [5, Corollary 5.4], that of coherent continuous domains [13, Theorem 2.10], and that of coherent quasicontinuous domains [9, Theorem 6.2]. In this paper, we will consider a special topological space, called a locally finitary compact space, and show that the probabilistic powerdomain of a coherent locally finitary compact T_0 space is coherent

☆ Research supported by NSF of China (No. 11371262).
* Corresponding author.
E-mail addresses: zhenchaolyu@sina.com (Z. Lyu), kouhui@scu.edu.cn (H. Kou).

https://doi.org/10.1016/j.topol.2018.01.008
0166-8641/

quasicontinuous. Specifically, from a topological viewpoint, we consider the weak topology of the probabilistic powerdomain and borrow a key tool from [3] to achieve our purpose. As a result, we obtain a new, short proof of Larrecq's and Jung's result [9]. Unfortunately, we still don't know whether the probabilistic powerdomain of a quasicontinuous domain is again quasicontinuous.

As we know, Claire Jones and Gordon Plotkin proved that the probabilistic powerdomain of a continuous domain is still a continuous domain in [5]. And Jones conjectured that the continuity condition is essential to make sure that the probabilistic powerdomain of a dcpo is continuous. In this paper we will confirm this conjecture and obtain some extra conclusions about the probabilistic powerdomain as well.

An outline of this paper is as follows. We devote section 2 to an introduction of the required notions and some techniques. In section 3 we prove that the probabilistic powerdomain of a coherent locally finitary compact T_0 space is also coherent and quasicontinuous. Using a similar method we obtain a new proof of Jones's result, which asserts that probabilistic powerdomain of a continuous domain endowed with the Scott topology is continuous. In the last section we try to get the property of L from its probabilistic powerdomain $\mathcal{V}(L)$.

2. Preliminaries

We assume some knowledge of basic domain theory and topology, as in, e.g., [1,2,7].

Let D be a dcpo. For $x, y \in D$, we say that $x \prec y$ iff whenever the intersection of a nonempty collection of upper sets is contained in $\uparrow y$, then the intersection of finitely many is contained in $\uparrow x$. D is hypercontinuous if $\{d \in D : d \prec x\}$ is directed and $x = \bigvee\{d \in D : d \prec x\}$ for all $x \in D$. For $x, y \in D$ when D is a complete lattice, we say that $x \lll y$, if for any subset A of D, $y \le \bigvee A$ implies $x \le a$ for some $a \in A$. D is prime continuous if $x = \bigvee\{d \in D : d \lll x\}$ for all $x \in D$.

For $F \subseteq D, x \in D$, we say that $F \ll x$ if for every directed set $A \subseteq D$, $\bigvee A \in \uparrow x$ implies $a \in \uparrow F$ for some $a \in A$. D is *quasicontinuous* if there is a directed subset A of $\{F \subseteq D : F \text{ is finite}, F \ll x\}$ with $\uparrow x = \bigcap\{\uparrow F : F \in A\}$ for all $x \in D$. In particular D will be called *continuous* if each element of A is a single point set.

Given a T_0 topological space (X, τ), its *specialisation order* is defined by $x \sqsubseteq y$ iff $x \in \overline{\{y\}}$. A T_0 space (X, τ) is called a *c-space* iff $\forall x \in U, x \in X, U \in \tau, \exists y \in U$ such that $x \in (\uparrow y)^\circ$. A T_0 space (X, τ) is called *locally finitary compact* iff $\forall x \in U, x \in X, U \in \tau$, there exists a finite subset F of U such that $x \in (\uparrow F)^\circ$.

A topological space is called *coherent* iff the intersection of any two compact saturated subsets is compact. A *stably locally compact* space is a sober, locally compact and coherent space. A topological space is called *stably compact* iff it is stably locally compact and compact.

We write $\sigma(D)$ for the Scott topology of D. In order to study quasicontinuous domains, we need the following lemma.

Lemma 2.1 *(Rudin's Lemma). Let $\mathcal{F}$ be a directed family of nonempty finite subsets of a partially ordered set P. Then there exists a directed set $D \subseteq \bigcup_{F\in\mathcal{F}} F$ such that $D \cap F \neq \varnothing$ for all $F \in \mathcal{F}$.*

Let (Y, τ) be a topological space and $\sqsubseteq$ its *specialisation order*. If $(Y, \sqsubseteq)$ is a dcpo, we denote the Scott topology on $(Y, \sqsubseteq)$ by $\sigma(Y)$. A natural question arises: when is τ equal to $\sigma(Y)$? Next we give an answer to this question.

Proposition 2.2. *Let (Y, τ) be a T_0 space such that $(Y, \sqsubseteq)$ is a dcpo and $\tau \subseteq \sigma(Y)$.*

1. *If (Y, τ) is a c-space, then $(Y, \sqsubseteq)$ is a continuous domain and $\tau = \sigma(Y)$.*
2. *If (Y, τ) is a locally finitary compact space, then $(Y, \sqsubseteq)$ is a quasicontinuous domain and $\tau = \sigma(Y)$.*